6 Topics in Analytic Geometry

6.2 Introduction to Conics: Parabolas

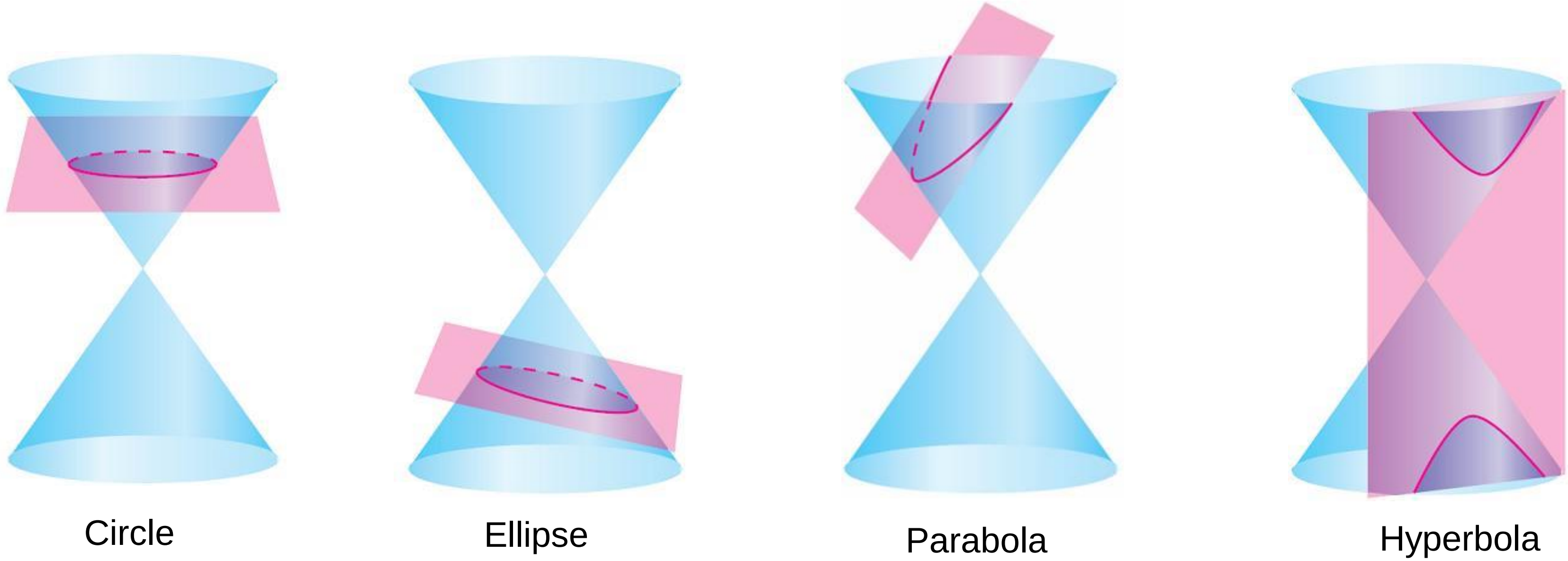
Objectives

- Recognize a conic as the intersection of a plane and a double-napped cone.
- Write equations of parabolas in standard form.
- Use the reflective property of parabolas to solve real-life problems.

Conics

Conics

A **conic section** (or simply **conic**) is the intersection of a plane and a double-napped cone. Notice in Figure 6.7 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone.



Basic Conics

Figure 6.7

Conics

When the plane does pass through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure 6.8.

Point Line Two Intersecting Lines

Degenerate Conics

Figure 6.8

Conics

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach, in which each of the conics is defined as a **locus** (collection) of points satisfying a geometric property.

Conics

For example, a circle as the set of all points (x, y) that are equidistant from a fixed point (h, k) led to the standard form of the equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2.$$

Equation of circle

Parabolas

Parabolas

You have learned that the graph of the quadratic function

$$f(x) = ax^2 + bx + c$$

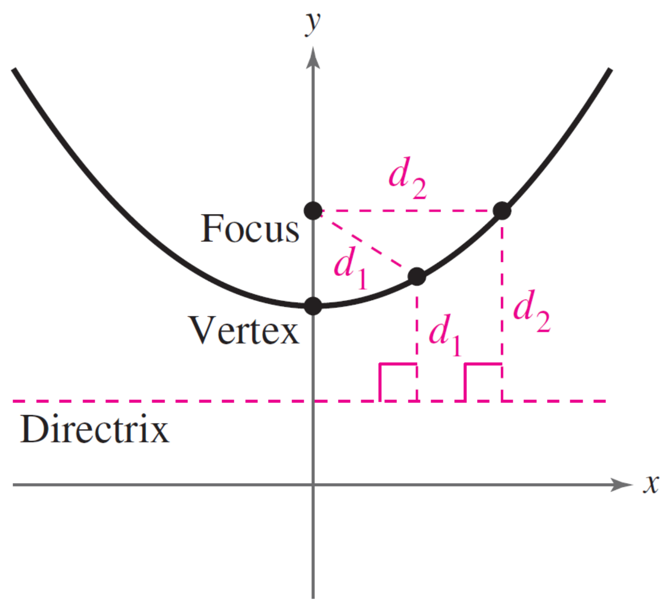
is a parabola that opens upward or downward.

Parabolas

The following definition of a parabola is more general in the sense that it is independent of the orientation of the parabola.

Definition of Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line, called the **directrix,** and a fixed point, called the **focus,** not on the line. (See figure.) The **vertex** is the midpoint between the focus and the directrix. The **axis** of the parabola is the line passing through the focus and the vertex.



Parabolas

Note that a parabola is symmetric with respect to its axis. Using the definition of a parabola, you can derive the following **standard form of the equation of a parabola** whose directrix is parallel to the *x*-axis or to the *y*-axis.

Standard Equation of a Parabola

The **standard form of the equation of a parabola** with vertex at (h, k) is as follows.

$(x - h)^2 = 4p(y - k), \quad p \neq 0$ — Vertical axis; directrix: $y = k - p$

$(y - k)^2 = 4p(x - h), \quad p \neq 0$ — Horizontal axis; directrix: $x = h - p$

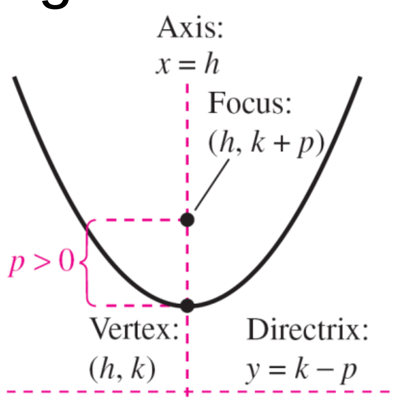
The focus lies on the axis p units (*directed distance*) from the vertex. If the vertex is at the origin, then the equation takes one of the following forms.

$x^2 = 4py$ — Vertical axis

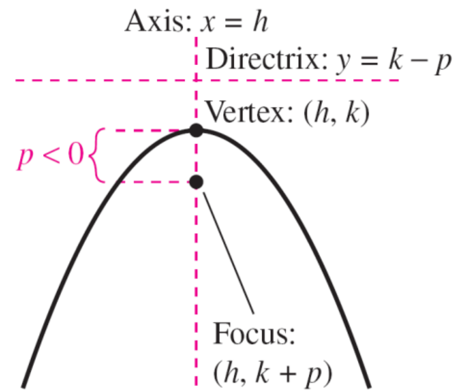
$y^2 = 4px$ — Horizontal axis

Parabolas

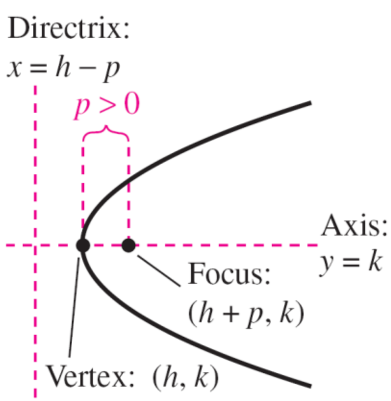
See the figures below.



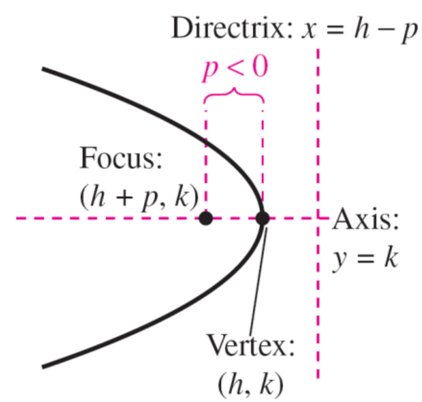
$(x – h)^2 = 4p(y – k)$
Vertical axis: $p > 0$



$(x – h)^2 = 4p(y – k)$
Vertical axis: $p < 0$



$(y – k)^2 = 4p(x – h)$
Horizontal axis: $p > 0$



$(y – k)^2 = 4p(x – h)$
Horizontal axis: $p < 0$

Example 1 – *Finding the Standard Equation of a Parabola*

Find the standard form of the equation of the parabola with vertex at the origin and focus (2, 0).

Solution:

The axis of the parabola is horizontal, passing through (0, 0) and (2, 0), as shown in Figure 6.9.

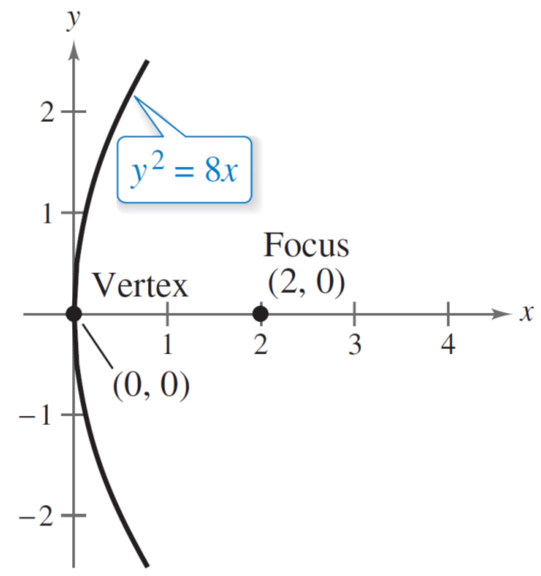


Figure 6.9

The standard form is $y^2 = 4px$, where $p = 2$. So, the equation is $y^2 = 8x$.

Example 1 – *Solution*

cont'd

You can use a graphing utility to confirm this equation.

To do this, let $y_1 = \sqrt{8x}$ to graph the upper portion and let $y_2 = -\sqrt{8x}$ to graph the lower portion of the parabola.

Application

Application

A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord.**

The specific focal chord perpendicular to the axis of the parabola is called the **latus rectum.**

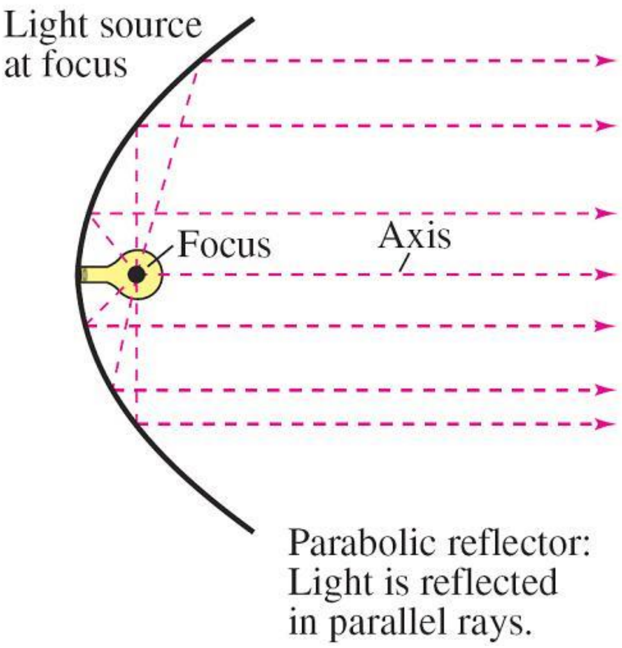
Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola about its axis.

The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola.

Application

This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes.

Conversely, the light rays emanating from the focus of a parabolic reflector used in a flashlight are all parallel to one another, as shown at the right.



Application

A line is **tangent** to a parabola at a point on the parabola when the line intersects, but does not cross, the parabola at the point.

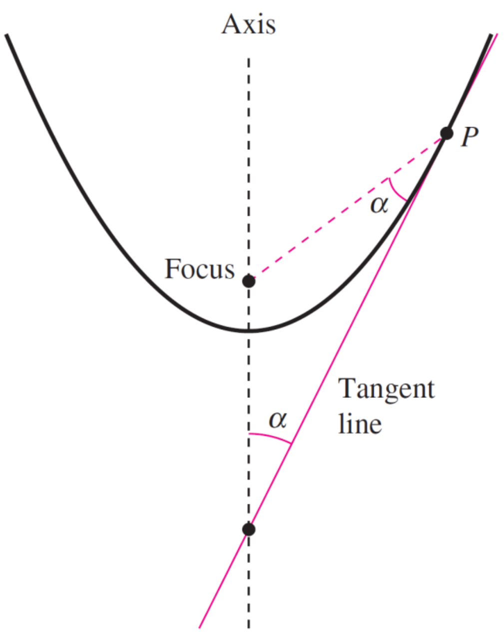
Tangent lines to parabolas have special properties related to the use of parabolas in constructing reflective surfaces.

Application

Reflective Property of a Parabola

The tangent line to a parabola at a point *P* makes equal angles with the following two lines (see figure below).

1. The line passing through *P* and the focus
2. The axis of the parabola

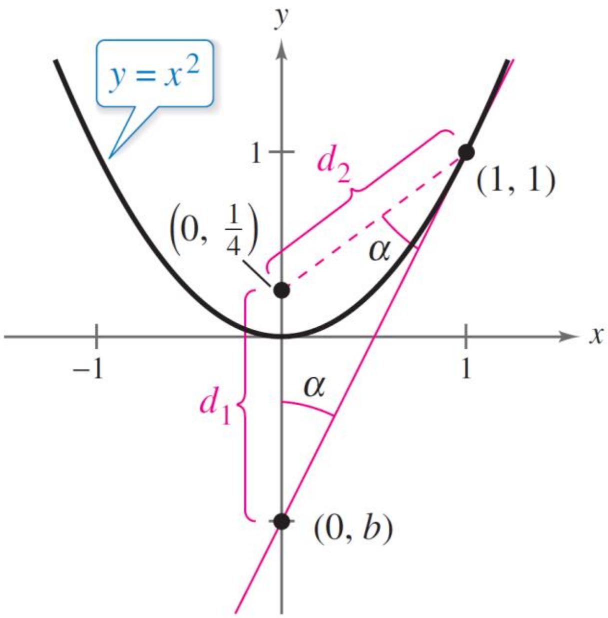


Example 4 – *Finding the Tangent Line at a Point on a Parabola*

Find the equation of the tangent line to the parabola $y = x^2$ at the point (1, 1).

Solution:

For this parabola, $p = \frac{1}{4}$ and the focus is $\left(0, \frac{1}{4}\right)$, as shown in figure below.



Example 4 – *Solution*

cont'd

You can find the *y*-intercept (0, *b*) of the tangent line by equating the lengths of the two sides of the isosceles triangle shown in the figure:

$$d_1 = \frac{1}{4} - b$$

and

$$d_2 = \sqrt{(1-0)^2 + \left[1 - \left(\frac{1}{4}\right)\right]^2} = \frac{5}{4}.$$

Note that $d_1 = \frac{1}{4} - b$ rather than $b - \frac{1}{4}$. The order of subtraction for the distance is important because the distance must be positive.

Example 4 – *Solution*

cont'd

Setting $d_1 = d_2$ produces

$$\frac{1}{4} - b = \frac{5}{4}$$

$$b = -1.$$

So, the slope of the tangent line is

$$m = \frac{1 - (-1)}{1 - 0} = 2$$

and the equation of the tangent line in slope-intercept form is

$$y = 2x - 1.$$