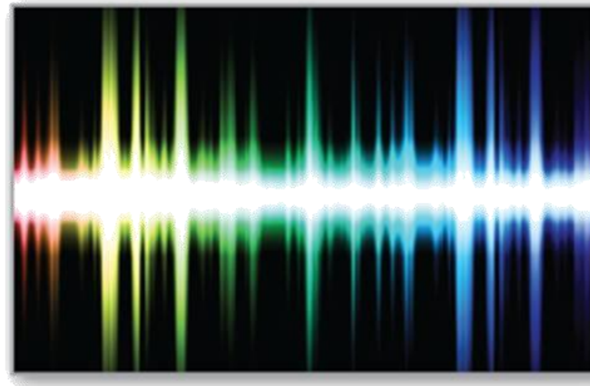
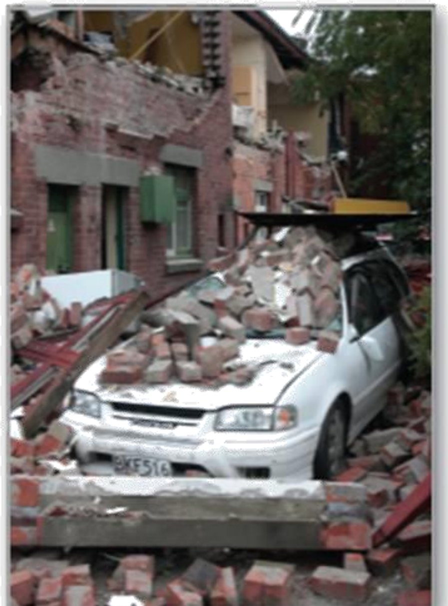
5 Exponential and Logarithmic Functions





5.5

Exponential and Logarithmic Models

Objectives

- Recognize the five most common types of models involving exponential and logarithmic functions.

- Use exponential growth and decay functions to model and solve real-life problems.

- Use Gaussian functions to model and solve real-life problems.

Objectives

- Use logistic growth functions to model and solve real-life problems.

- Use logarithmic functions to model and solve real-life problems.

Introduction

Introduction

The five most common types of mathematical models involving exponential functions and logarithmic functions are as follows.

1. Exponential growth model: $y = ae^{bx}$, $b > 0$

2. Exponential decay model: $y = ae^{-bx}$, $b > 0$

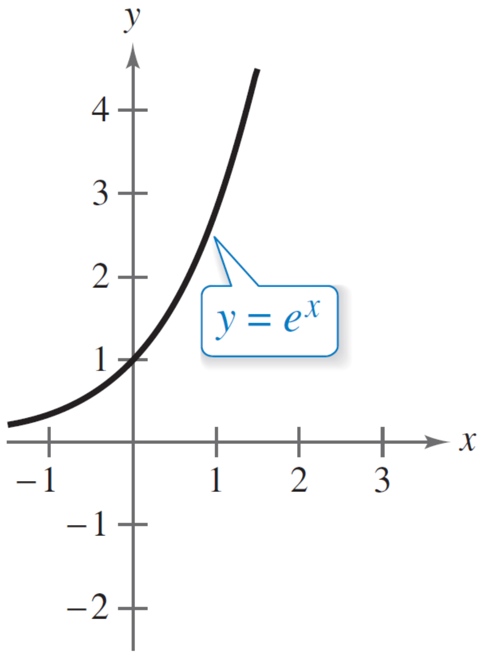
3. Gaussian model: $y = ae^{-(x - b)^2/c}$

4. Logistic growth model: $y = \dfrac{a}{1 + be^{-rx}}$

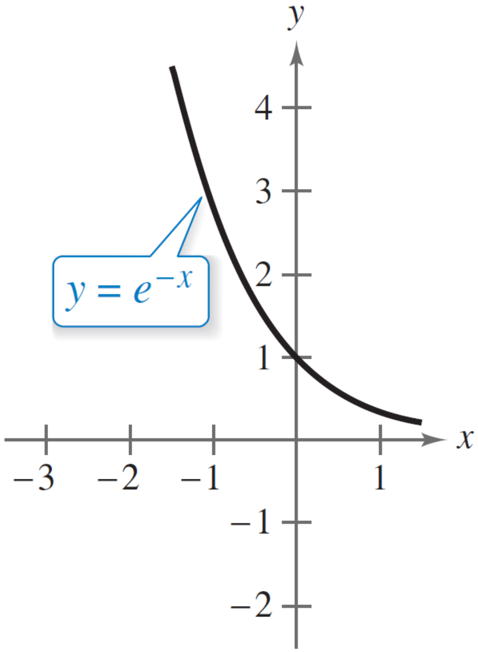
5. Logarithmic models: $y = a + b \ln x$, $y = a + b \log x$

Introduction

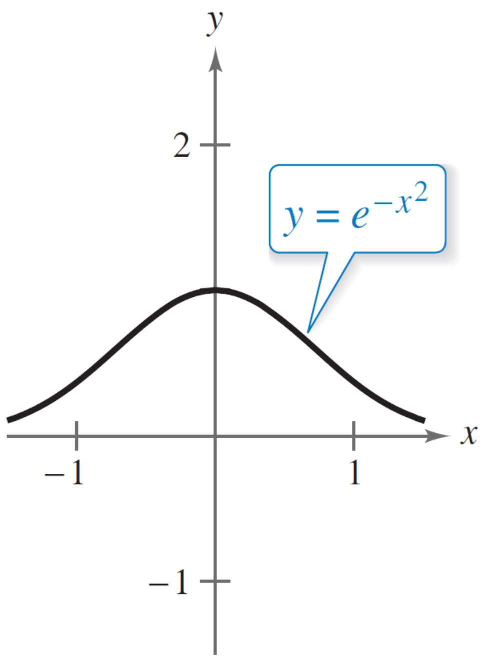
The basic shapes of the graphs of these functions are as follows.



Exponential growth model



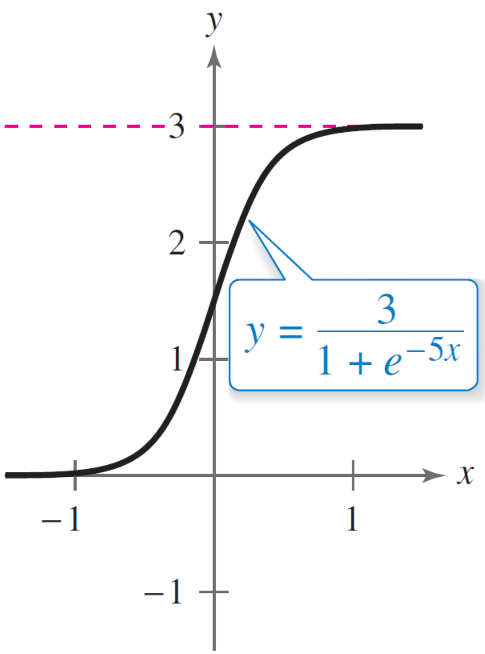
Exponential decay model



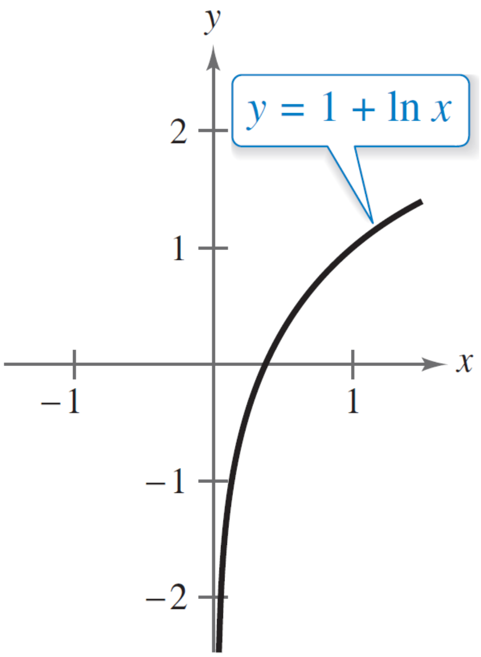
Gaussian model

Introduction

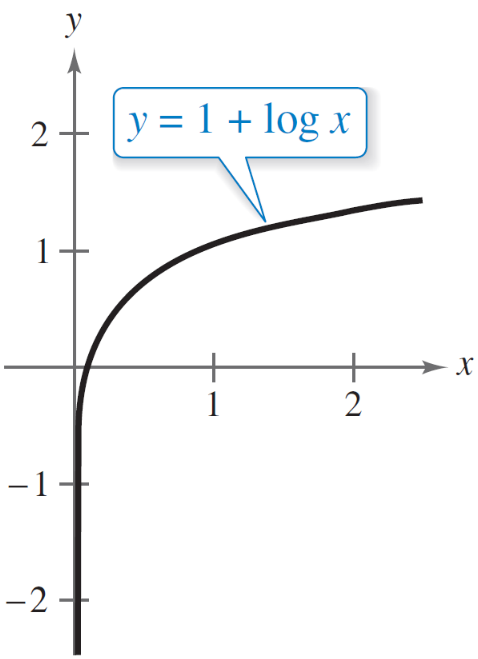
cont'd



Logistic growth model



Natural logarithmic model



Common logarithmic model

Introduction

You often gain insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the asymptotes of the graph of the function.

Exponential Growth and Decay

Example 1 – *Online Advertising*

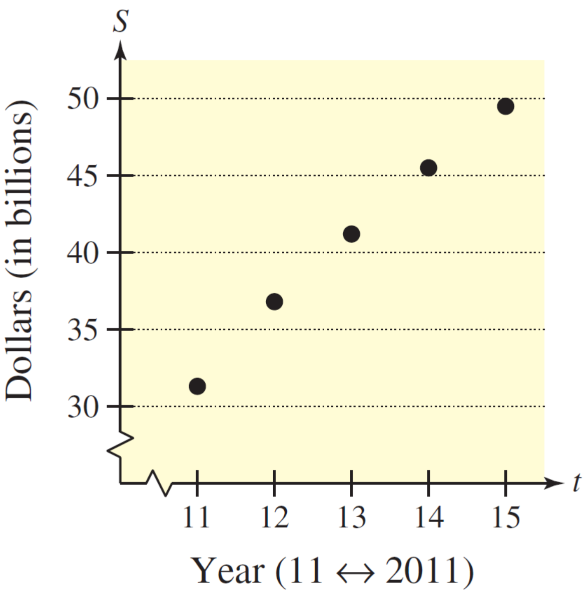
Estimates of the amounts (in billions of dollars) of U.S. online advertising spending from 2011 through 2015 are shown in the table.

Year	2011	2012	2013	2014	2015
Advertising Spending	31.3	36.8	41.2	45.5	49.5

Example 1 – *Online Advertising* cont'd

A scatter plot of the data is shown below.

An exponential growth model that approximates the data is given by $S = 9.30e^{0.1129t}$, $11 \leq t \leq 15$ where S is the amount of spending (in billions of dollars) and $t = 11$ represents 2011.



Online Advertising Spending

Example 1 – *Online Advertising* cont'd

Compare the values given by the model with the estimates shown in the table. According to this model, when will the amount of U.S. online advertising spending reach $80 billion?

Solution:

The following table compares the two sets of advertising spending amounts.

Year	2011	2012	2013	2014	2015
Advertising Spending	31.3	36.8	41.2	45.5	49.5
Model	32.2	36.0	40.4	45.2	50.6

Example 1 – *Solution*

cont'd

To find when the amount of U.S. online advertising spending will reach \$80 billion, let $S = 80$ in the model and solve for t.

$$9.30e^{0.1129t} = S$$ Write original model.

$$9.30e^{0.1129t} = 80$$ Substitute 80 for S.

$$e^{0.1129t} \approx 8.6022$$ Divide each side by 9.30.

$$\ln e^{0.1129t} \approx \ln 8.6022$$ Take natural log of each side.

$$0.1129t \approx 2.1520$$ Inverse Property

Example 1 – *Solution*

cont'd

$$t \approx 19.1$$ Divide each side by 0.1129.

According to the model, the amount of U.S. online advertising spending will reach $80 billion in 2019.

Exponential Growth and Decay

In living organic material, the ratio of the number of radioactive carbon isotopes (carbon 14) to the number of nonradioactive carbon isotopes (carbon 12) is about 1 to 10^{12}.

When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of about 5700 years.

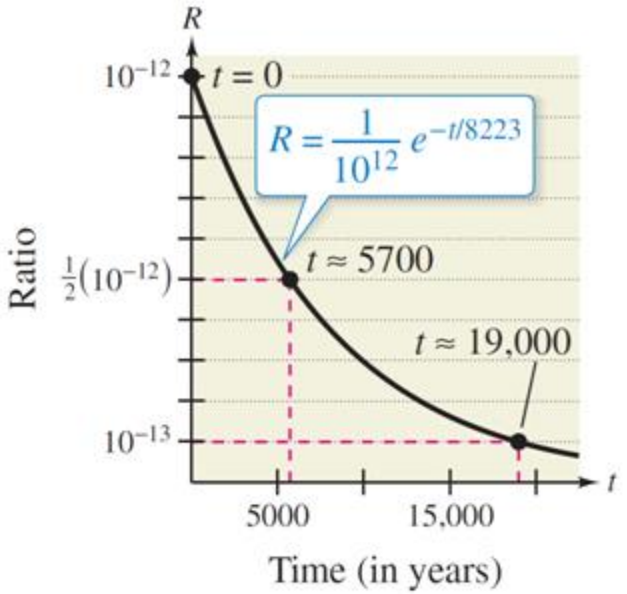
Exponential Growth and Decay

To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio of carbon 14 to carbon 12 present at any time *t* (in years).

$$R = \frac{1}{10^{12}} e^{-t/8223}$$

The graph of *R* is shown at the right. Note that *R* decreases as *t* increases.

Carbon dating model



Carbon dating model

Exponential Growth and Decay

The value of b in the exponential decay model $y = ae^{-bt}$ determines the *decay* of radioactive isotopes.

For instance, to find how much of an initial 10 grams of ^{226}Ra isotope with a half-life of 1599 years is left after 500 years, substitute this information into the model $y = ae^{-bt}$.

$$\frac{1}{2}(10) = 10e^{-b(1599)} \quad \Rightarrow \quad \ln\frac{1}{2} = -1599b \quad \Rightarrow \quad b = -\frac{\ln\frac{1}{2}}{1599}$$

Using the value of b found above and $a = 10$, the amount left is

$$y = 10e^{-[-\ln(1/2)/1599](500)} \approx 8.05 \text{ grams.}$$

Gaussian Models

Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

In probability and statistics, this type of model is commonly represents populations that are **normally distributed**. The graph of a Gaussian model is called a **bell-shaped curve**.

Try graphing the normal distribution with a graphing utility. Can you see why it is called a bell-shaped curve?

Gaussian Models

For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

The **average value** of a population can be found from the bell-shaped curve by observing where the maximum y–value of the function occurs.

The x–value corresponding to the maximum y–value of the function represents the average value of the independent variable—in this case, x.

Example 4 – *SAT Scores*

In 2011, the SAT mathematics scores for high school graduates in the United States roughly followed the normal distribution given by

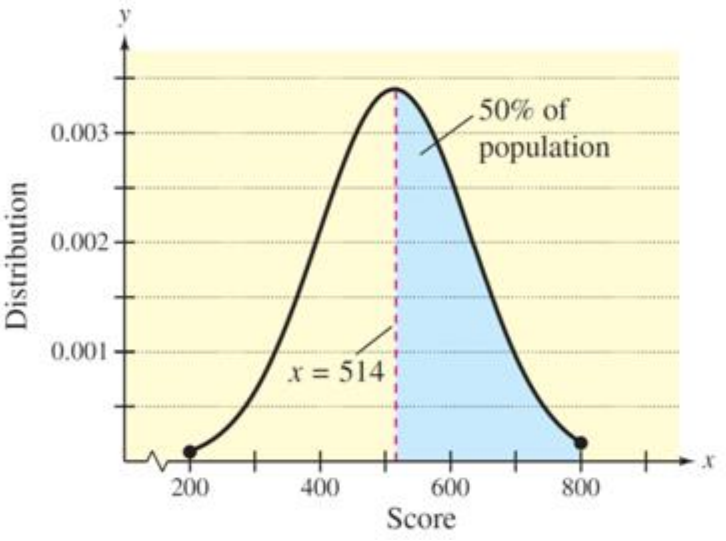
$$y = 0.0034e^{-(x-514)^2/27{,}378}, \qquad 200 \le x \le 800$$

where x is the SAT score for mathematics. Sketch the graph of this function. From the graph, estimate the average SAT mathematics score.

Example 4 – *Solution*

The graph of the function is shown below. On this bell-shaped curve, the maximum value of the curve represents the average score.

From the graph, you can estimate that the average mathematics Score for college-bound seniors in 2011 was 514.



SAT Scores

Logistic Growth Models

Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as indicated by the graph in Figure 5.21.

One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where *y* is the population size and *x* is the time.

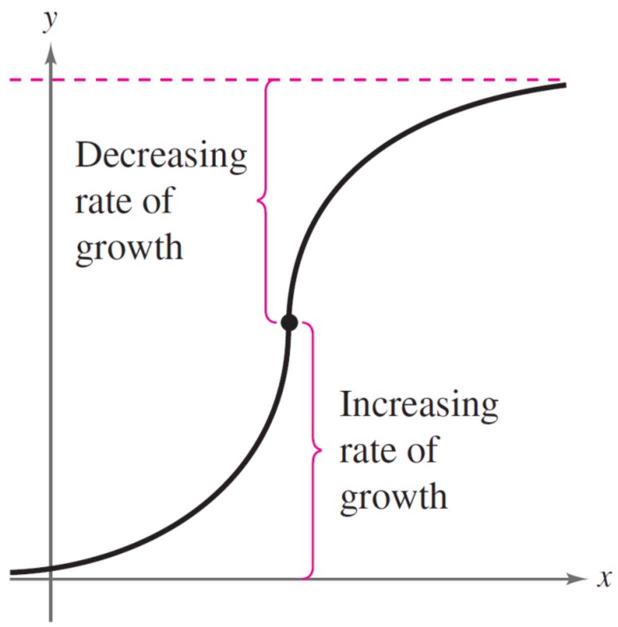


Figure 5.21

Logistic Growth Models

An example is a bacteria culture that is initially allowed to grow under ideal conditions, and then under less favorable conditions that inhibit growth.

A logistic growth curve is also called a **sigmoidal curve**.

Example 5 – *Spread of a Virus*

On a college campus of 5000 students, one student returns from vacation with a contagious and long-lasting flu virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \qquad t \geq 0$$

where *y* is the total number of students infected after *t* days. The college will cancel classes when 40% or more of the students are infected.

a. How many students are infected after 5 days?

b. After how many days will the college cancel classes?

Example 5(a) – *Solution*

After 5 days, the number of students infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}}$$

$$= \frac{5000}{1 + 4999e^{-4}}$$

$$\approx 54.$$

Example 5(b) – *Solution*

cont'd

The college will cancel classes when the number of infected students is (0.40)(5000) = 2000.

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$e^{-0.8t} = \frac{1.5}{4999}$$

Example 5(b) – *Solution*

cont'd

$$-0.8t = \ln \frac{1.5}{4999}$$

$$t = -\frac{1}{0.8} \ln \frac{1.5}{4999}$$

$$t \approx 10.1$$

So, after about 10 days, at least 40% of the students will be infected, and the college will cancel classes.

Logarithmic Models

Example 6 – *Magnitudes of Earthquakes*

On the Richter scale, the magnitude R of an earthquake of intensity I is given by

$$R = \log \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensity of each earthquake. (Intensity is a measure of the wave energy of an earthquake.)

a. Alaska in 2012: $R = 4.0$

b. Christchurch, New Zealand, in 2011: $R = 6.3$

Example 6 – *Solution*

a. Because $I_0 = 1$ and $R = 4.0$, you have

$$4.0 = \log \frac{I}{1}$$ Substitute 1 for I_0 and 4.0 for R.

$$10^{4.0} = 10^{\log I}$$ Exponentiate each side.

$$10^{4.0} = I$$ Inverse Property

$$10{,}000 = I$$ Simplify.

Example 6 – *Solution* cont'd

b. For R = 6.3, you have

$$6.3 = \log \frac{I}{1}$$ Substitute 1 for I_0 and 6.3 for R.

$$10^{6.3} = 10^{\log I}$$ Exponentiate each side.

$$10^{6.3} = I$$ Inverse Property

$$2{,}000{,}000 \approx I$$ Use a calculator.

Example 6 – *Solution*

cont'd

Note that an increase of 2.3 units on the Richter scale (from 4.0 to 6.3) represents an increase in intensity by a factor of 2,000,000/10,000 = 200.

In other words, the intensity of the earthquake in Christchurch was about 200 times as great as that of the earthquake in Alaska.