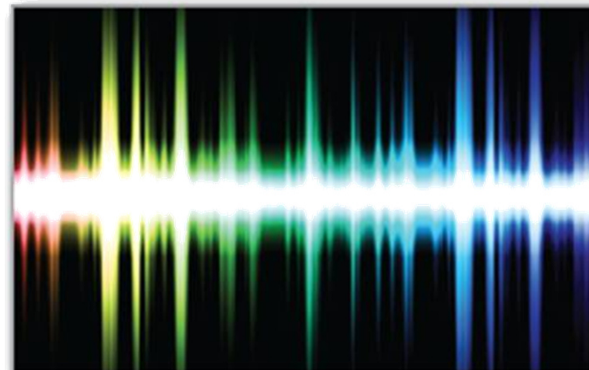
5 Exponential and Logarithmic Functions



5.4 Exponential and Logarithmic Equations

Objectives

- Solve simple exponential and logarithmic equations.
- Solve more complicated exponential equations.
- Solve more complicated logarithmic equations.
- Use exponential and logarithmic equations to model and solve real-life problems.

Introduction

Introduction

There are two basic strategies for solving exponential or logarithmic equations.

The first is based on the One-to-One Properties and was used to solve simple exponential and logarithmic equations.

The second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the following properties are true for all x and y for which $\log_a x$ and $\log_a y$ are defined.

Introduction

One-to-One Properties

$a^x = a^y$ if and only if $x = y$.

$\log_a x = \log_a y$ if and only if $x = y$.

Inverse Properties

$$a^{\log_a x} = x$$

$$\log_a a^x = x$$

Example 1 – *Solving Simple Equations*

Original Equation	Rewritten Equation	Solution	Property
a. $2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b. $\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
c. $\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
d. $e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
e. $\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
f. $\log x = -1$	$10^{\log x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse
g. $\log_3 x = 4$	$3^{\log_3 x} = 3^4$	$x = 81$	Inverse

Introduction

The strategies used in Example 1 are summarized as follows.

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Solving Exponential Equations

Example 2 – *Solving Exponential Equations*

Solve each equation and approximate the result to three decimal places, if necessary.

a. $e^{-x^2} = e^{-3x-4}$

b. $3(2^x) = 42$

Example 2(a) – *Solution*

$$e^{-x^2} = e^{-3x-4}$$ Write original equation.

$$-x^2 = -3x - 4$$ One-to-One Property

$$x^2 - 3x - 4 = 0$$ Write in general form.

$$(x + 1)(x - 4) = 0$$ Factor.

$$(x + 1) = 0 \Rightarrow x = -1$$ Set 1st factor equal to 0.

$$(x - 4) = 0 \Rightarrow x = 4$$ Set 2nd factor equal to 0.

The solutions are $x = -1$ and $x = 4$. Check these in the original equation.

Example 2(b) – *Solution*

cont'd

$3(2^x) = 42$	Write original equation.
$2^x = 14$	Divide each side by 3.
$\log_2 2^x = \log_2 14$	Take log (base 2) of each side.
$x = \log_2 14$	Inverse Property
$x = \dfrac{\ln 14}{\ln 2}$	Change-of-base formula
≈ 3.807	

The solution is $x = \log_2 14 \approx 3.807$. Check this in the original equation.

Solving Logarithmic Equations

Solving Logarithmic Equations

To solve a logarithmic equation, you can write it in exponential form.

$\ln x = 3$	Logarithmic form
$e^{\ln x} = e^3$	Exponentiate each side.
$x = e^3$	Exponential form

This procedure is called *exponentiating* each side of an equation.

Example 6 – *Solving Logarithmic Equations*

a. $\ln x = 2$ — Original equation

$e^{\ln x} = e^2$ — Exponentiate each side.

$x = e^2$ — Inverse Property

b. $\log_3(5x - 1) = \log_3(x + 7)$ — Original equation

$5x - 1 = x + 7$ — One-to-One Property

$x = 2$ — Solution

Example 6 – *Solving Logarithmic Equations* cont'd

c. $\log_6(3x + 14) - \log_6 5 = \log_6 2x$ — Original equation

$$\log_6\left(\frac{3x + 14}{5}\right) = \log_6 2x$$ Quotient Property of Logarithms

$$\frac{3x + 14}{5} = 2x$$ One-to-One Property

$$3x + 14 = 10x$$ Multiply each side by 5.

$$x = 2$$ Solution

Applications

Example 10 – *Doubling an Investment*

You invest $500 at an annual interest rate of 6.75%, compounded continuously. How long will it take your money to double?

Solution:

Using the formula for continuous compounding, the balance is

$$A = Pe^{rt}$$

$$A = 500e^{0.0675t}.$$

Example 10 – *Solution*

cont'd

To find the time required for the balance to double, let $A = 1000$ and solve the resulting equation for t.

$500e^{0.0675t} = 1000$ — Let $A = 1000$.

$e^{0.0675t} = 2$ — Divide each side by 500.

$\ln e^{0.0675t} = \ln 2$ — Take natural log of each side.

$0.0675t = \ln 2$ — Inverse Property

$t = \dfrac{\ln 2}{0.0675}$ — Divide each side by 0.0675.

$t \approx 10.27$ — Use a calculator.

Example 10 – *Solution*

cont'd

The balance in the account will double after approximately 10.27 years. This result is demonstrated graphically below.