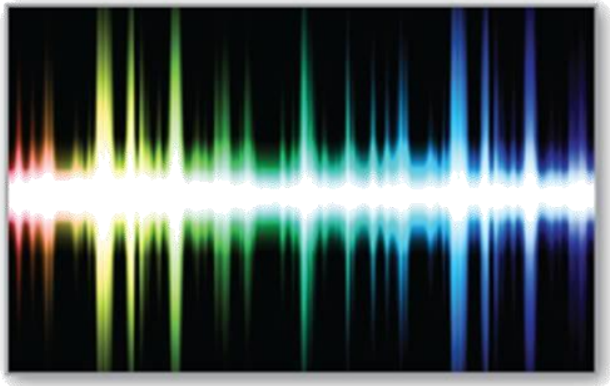
5 Exponential and Logarithmic Functions



5.3 Properties of Logarithms

Objectives

- Use the change-of-base formula to rewrite and evaluate logarithmic expressions.
- Use properties of logarithms to evaluate or rewrite logarithmic expressions.
- Use properties of logarithms to expand or condense logarithmic expressions.
- Use logarithmic functions to model and solve real-life problems.

Change of Base

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base *e*).

Although common logarithms and natural logarithms are the most frequently used, you may occasionally need to evaluate logarithms with other bases. To do this, use the following **change-of-base formula**.

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows.

Base *b*

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Base 10

$$\log_a x = \frac{\log x}{\log a}$$

Base *e*

$$\log_a x = \frac{\ln x}{\ln a}$$

Change of Base

One way to look at the change-of-base formula is that logarithms with base a are *constant multiples* of logarithms with base b. The constant multiplier is $\frac{1}{\log_b a}$.

Example 1 – *Changing Bases Using Common Logarithms*

$$\log_4 25 = \frac{\log 25}{\log 4}$$

$\log_a x = \dfrac{\log x}{\log a}$

$$\approx \frac{1.39794}{0.60206}$$

Use a calculator.

$$\approx 2.3219$$

Simplify.

Properties of Logarithms

Properties of Logarithms

You know from the preceding section that the logarithmic function with base *a* is the *inverse function* of the exponential function with base *a*.

So, it makes sense that the properties of exponents should have corresponding properties involving logarithms.

For instance, the exponential property $a^u a^v = a^{u+v}$ has the corresponding logarithmic property $\log_a(uv) = \log_a u + \log_a v$.

Properties of Logarithms

Properties of Logarithms

Let a be a positive number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

	Logarithm with Base a	Natural Logarithm
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

Example 3 – *Using Properties of Logarithms*

Write each logarithm in terms of ln 2 and ln 3.

a. ln 6 **b.** $\ln \frac{2}{27}$

Solution:

a. $\ln 6 = \ln (2 \cdot 3)$ Rewrite 6 as $2 \cdot 3$.

$= \ln 2 + \ln 3$ Product Property

b. $\ln \frac{2}{27} = \ln 2 - \ln 27$ Quotient Property

$= \ln 2 - \ln 3^3$ Rewrite 27 as 3^3.

$= \ln 2 - 3 \ln 3$ Power Property

Rewriting Logarithmic Expressions

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra.

This is true because these properties convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

Example 5 – *Expanding Logarithmic Expressions*

Expand each logarithmic expression.

a. $\log_4 5x^3y$

b. $\ln \dfrac{\sqrt{3x - 5}}{7}$

Solution:

a. $\log_4 5x^3y = \log_4 5 + \log_4 x^3 + \log_4 y$ Product Property

$= \log_4 5 + 3 \log_4 x + \log_4 y$ Power Property

Example 5 – *Solution*

cont'd

b. $\ln \dfrac{\sqrt{3x - 5}}{7} = \ln \dfrac{(3x - 5)^{1/2}}{7}$ Rewrite using rational exponent.

$= \ln(3x - 5)^{1/2} - \ln 7$ Quotient Property

$= \dfrac{1}{2}\ln(3x - 5) - \ln 7$ Power Property

Application

Application

One method of determining how the x- and y-values for a set of nonlinear data are related is to take the natural logarithm of each of the x- and y-values.

If the points, when graphed, fall on a line, then you can determine that the x- and y-values are related by the equation

$$\ln y = m \ln x$$

where m is the slope of the line.

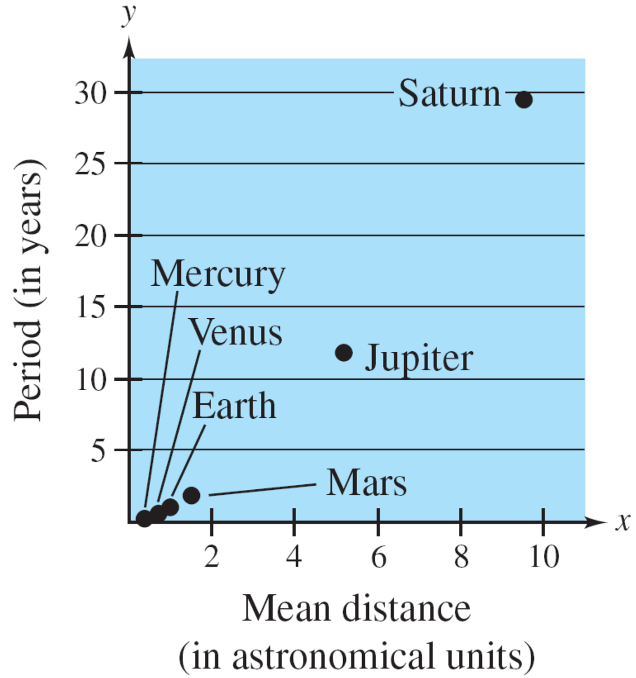
Example 7 – *Finding a Mathematical Model*

The table shows the mean distance from the sun *x* and the period *y* (the time it takes a planet to orbit the sun) for each of the six planets that are closest to the sun. In the table, the mean distance is given in terms of astronomical units (where Earth's mean distance is defined as 1.0), and the period is given in years. Find an equation that relates *y* and *x*.

Planet	Mean Distance, *x*	Period, *y*
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.860
Saturn	9.555	29.420

Example 7 – *Solution*

Figure 5.18 shows the plots of the points given by the above table. From this figure, it is not clear how to find an equation that relates *y* and *x*.



Planets Near the Sun

Figure 5.18

Example 7 – *Solution*

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To solve this problem, take the natural logarithm of each of the *x*- and *y*-values, as shown in the table below.

Planet	ln x	ln y
Mercury	-0.949	-1.423
Venus	-0.324	-0.486
Earth	0.000	0.000
Mars	0.421	0.632
Jupiter	1.649	2.473
Saturn	2.257	3.382

Example 7 – *Solution*

cont'd

Now, by plotting the points in the table, you can see that all six of the points appear to lie in a line (see Figure 5.19). Choose any two points to determine the slope of the line.

Using the points (0.421, 0.632) and (0, 0), the slope of the line is

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}.$$

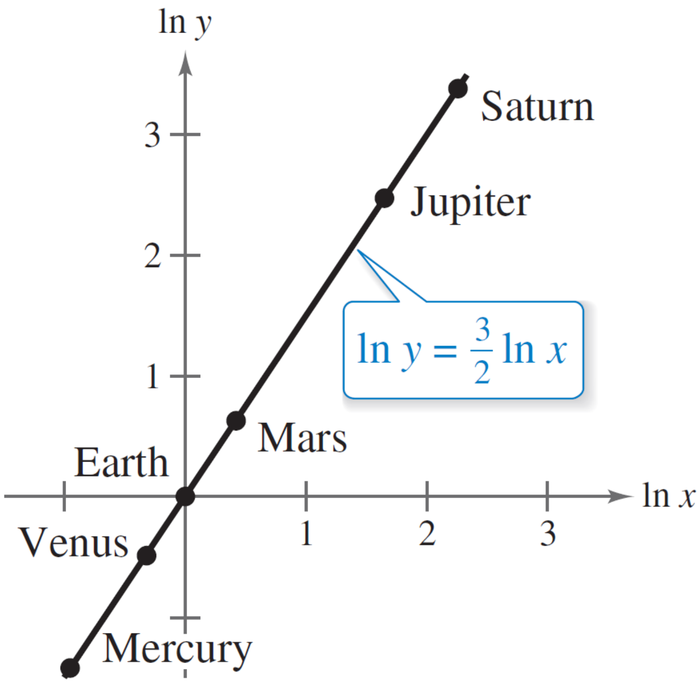

Figure 5.19

Example 7 – *Solution*

cont'd

By the point-slope form, the equation of the line is $Y = \frac{3}{2}X$, where $Y = \ln y$ and $X = \ln x$.

So, $\ln y = \frac{3}{2} \ln x$.