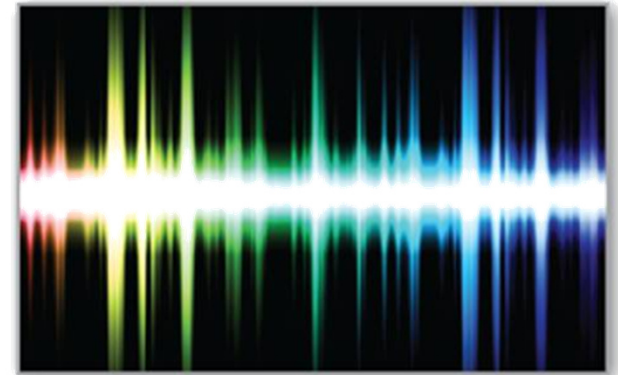


5 Exponential and Logarithmic Functions



5.2

Logarithmic Functions and Their Graphs

Objectives

- Recognize and evaluate logarithmic functions with base a .
- Graph logarithmic functions.
- Recognize, evaluate, and graph natural logarithmic functions.
- Use logarithmic functions to model and solve real-life problems.



Logarithmic Functions

Logarithmic Functions

Every function of the form $f(x) = a^x$ passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base a** .

Definition of Logarithmic Function with Base a

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \text{ if and only if } x = a^y.$$

The function

$$f(x) = \log_a x \quad \text{Read as “log base } a \text{ of } x.”$$

is called the **logarithmic function with base a** .

Logarithmic Functions

The equations

$$y = \log_a x \text{ and } x = a^y$$

are equivalent. The first equation is in logarithmic form and the second is in exponential form.

For example, $2 = \log_3 9$ is equivalent to $9 = 3^2$, and $5^3 = 125$ is equivalent to $\log_5 125 = 3$.

Logarithmic Functions

When evaluating logarithms, remember that *a logarithm is an exponent*. This means that $\log_a x$ is the exponent to which *a* must be raised to obtain *x*.

For instance, $\log_2 8 = 3$ because 2 raised to the third power is 8.

Example 1 – *Evaluating Logarithms*

Evaluate each logarithm at the indicated value of x .

a. $f(x) = \log_2 x$, $x = 32$

b. $f(x) = \log_3 x$, $x = 1$

c. $f(x) = \log_4 x$, $x = 2$

d. $f(x) = \log_{10} x$, $x = \frac{1}{100}$

Solution:

a. $f(32) = \log_2 32$

because $2^5 = 32$.

$$= 5$$

b. $f(1) = \log_3 1$

because $3^0 = 1$.

$$= 0$$

Example 1 – *Solution*

cont'd

c. $f(2) = \log_4 2$

$$= \frac{1}{2}$$

because $4^{1/2} = \sqrt{4} = 2$.

d. $f\left(\frac{1}{100}\right) = \log_{10} \frac{1}{100}$

$$= -2$$

because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

Logarithmic Functions

The logarithmic function with base 10 is called the **common logarithmic function**. It is denoted by \log_{10} or simply by \log . On most calculators, this function is denoted by $\boxed{\text{LOG}}$.

The following properties follow directly from the definition of the logarithmic function with base a .

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.
2. $\log_a a = 1$ because $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$
4. If $\log_a x = \log_a y$, then $x = y$.

Inverse Properties

One-to-One Property



Graphs of Logarithmic Functions

Graphs of Logarithmic Functions

To sketch the graph of $y = \log_a x$, use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

Example 5 – *Graphs of Exponential and Logarithmic Functions*

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$

b. $g(x) = \log_2 x$

Example 5(a) – *Solution*

For $f(x) = 2^x$, construct a table of values.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

By plotting these points and connecting them with a smooth curve, you obtain the graph shown in Figure 5.10.

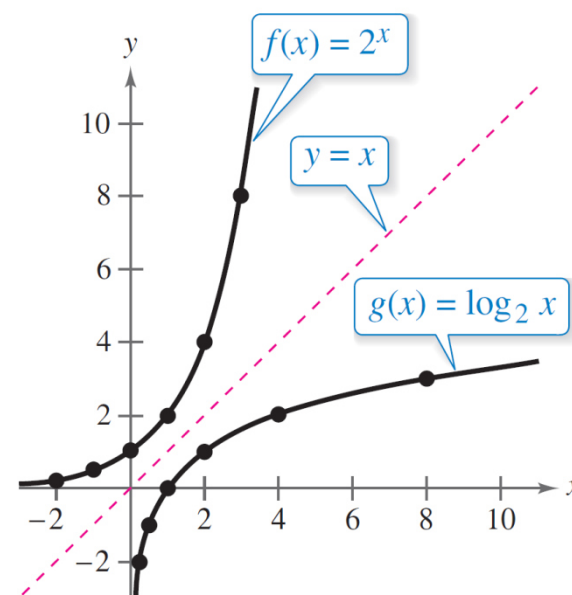


Figure 5.10

Example 5(b) – *Solution*

cont'd

Because $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve.

The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 5.10.

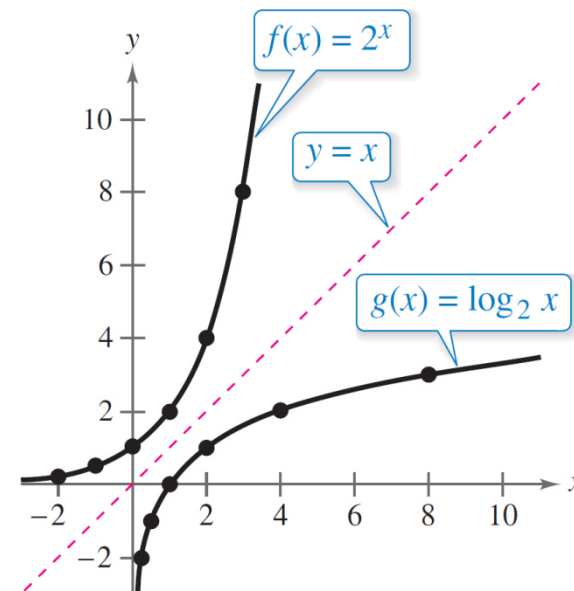


Figure 5.10

Graphs of Logarithmic Functions

The nature of the graph in Figure 5.11 is typical of functions of the form $f(x) = \log_a x$, $a > 1$. They have one x -intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$.

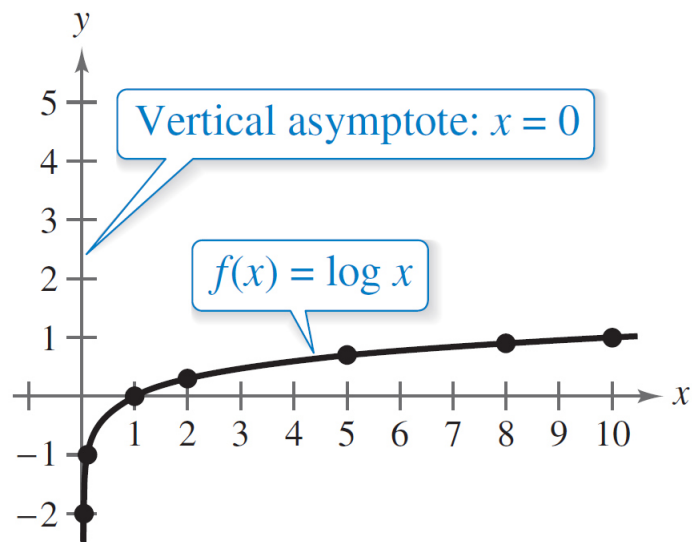


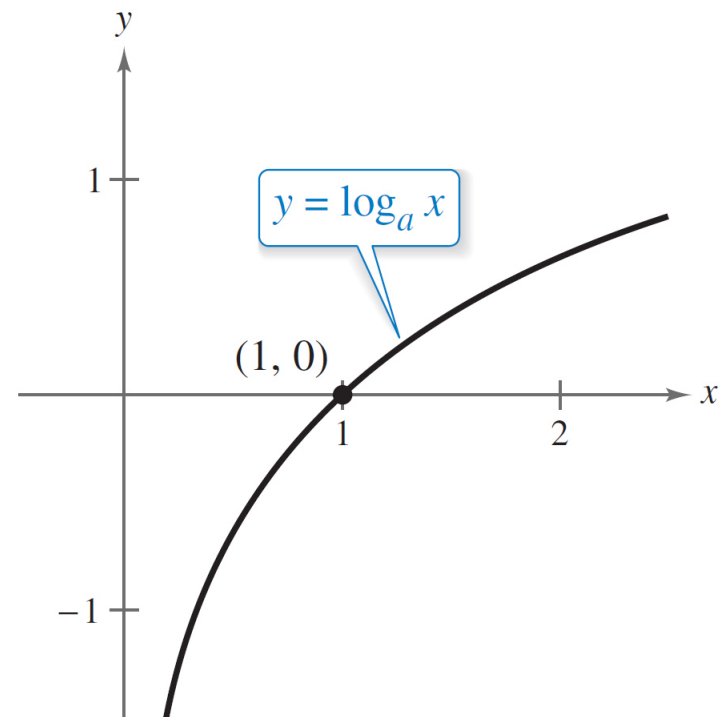
Figure 5.11

Graphs of Logarithmic Functions

The following summarizes the basic characteristics of logarithmic graphs.

Graph of $y = \log_a x$, $a > 1$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- x -intercept: $(1, 0)$
- Increasing
- One-to-one, therefore has an inverse function



Graphs of Logarithmic Functions

- y -axis is a vertical asymptote ($\log_a x \rightarrow -\infty$ as $x \rightarrow 0^+$).
- Continuous
- Reflection of graph of $y = a^x$ about the line $y = x$.

The basic characteristics of the graph of $f(x) = a^x$ are shown below to illustrate the inverse relation between $f(x) = a^x$ and $g(x) = \log_a x$.

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y -intercept: $(0, 1)$
- x -axis is a horizontal asymptote ($a^x \rightarrow 0$ as $x \rightarrow -\infty$).



The Natural Logarithmic Function

The Natural Logarithmic Function

We will see that $f(x) = e^x$ is one-to-one and so has an inverse function.

This inverse function is called the **natural logarithmic function** and is denoted by the special symbol $\ln x$, read as “the natural log of x ” or “el en of x .” Note that the natural logarithm is written without a base. The base is understood to be e .

The Natural Logarithmic Function

The function defined by

$$f(x) = \log_e x = \ln x, \quad x > 0$$

is called the **natural logarithmic function**.

The Natural Logarithmic Function

The above definition implies that the natural logarithmic function and the natural exponential function are inverse functions of each other.

So, every logarithmic equation can be written in an equivalent exponential form, and every exponential equation can be written in an equivalent logarithmic form.

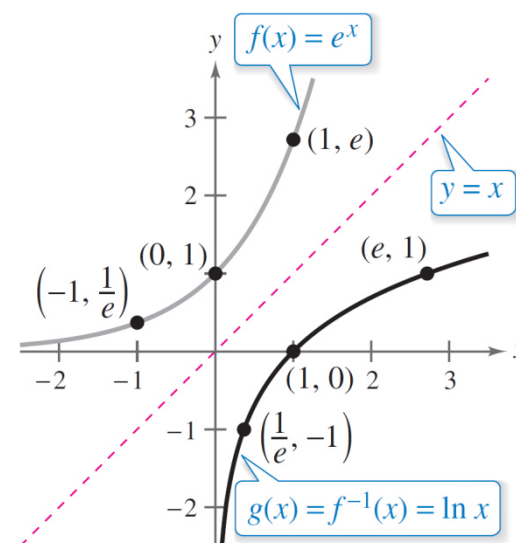
That is, $y = \ln x$ and $x = e^y$ are equivalent equations.

The Natural Logarithmic Function

Because the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions of each other, their graphs are reflections of each other in the line $y = x$.

Figure 5.14 illustrates this reflective property.

On most calculators, $\boxed{\text{LN}}$ denotes the natural logarithm, as illustrated in Example 8.



Reflection of graph of $f(x) = e^x$ in the line $y = x$.

Figure 5.14

Example 8 – *Evaluating the Natural Logarithmic Function*

Use a calculator to evaluate the function $f(x) = \ln x$ at each value of x .

a. $x = 2$

b. $x = 0.3$

c. $x = -1$

d. $x = 1 + \sqrt{2}$

Example 8 – *Solution*

Function Value	Graphing Calculator Keystrokes	Display
a. $f(2) = \ln 2$	$\boxed{\text{LN}} \ 2 \ \boxed{\text{ENTER}}$	0.6931472
b. $f(0.3) = \ln 0.3$	$\boxed{\text{LN}} \ .3 \ \boxed{\text{ENTER}}$	-1.2039728
c. $f(-1) = \ln(-1)$	$\boxed{\text{LN}} \ \boxed{(-)} \ 1 \ \boxed{\text{ENTER}}$	ERROR
d. $f(1 + \sqrt{2}) = \ln(1 + \sqrt{2})$	$\boxed{\text{LN}} \ \boxed{(} \ 1 \ \boxed{+} \ \boxed{\sqrt{}} \ 2 \ \boxed{)} \ \boxed{\text{ENTER}}$	0.8813736

The Natural Logarithmic Function

The four properties of logarithms are also valid for natural logarithms.

Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.

2. $\ln e = 1$ because $e^1 = e$.

3. $\ln e^x = x$ and $e^{\ln x} = x$

Inverse Properties

4. If $\ln x = \ln y$, then $x = y$.

One-to-One Property



Application

Example 11 – *Human Memory Model*

Students participating in a psychology experiment attended several lectures on a subject and took an exam.

Every month for a year after the exam, the students took retest to see how much of the material they remembered.

The average scores for the group are given by the *human memory model* $f(t) = 75 - 6 \ln(t + 1)$, $0 \leq t \leq 12$, where t is the time in months.

Example 11 – *Human Memory Model* cont'd

- a.** What was the average score on the original exam ($t = 0$)?
- b.** What was the average score at the end of $t = 2$ months?
- c.** What was the average score at the end of $t = 6$ months?

Example 11(a) – *Solution*

The original average score was

$$f(0) = 75 - 6 \ln(0 + 1)$$

Substitute 0 for t .

$$= 75 - 6 \ln 1$$

Simplify.

$$= 75 - 6(0)$$

Property of natural logarithms

$$= 75.$$

Solution

Example 11(b) – *Solution*

cont'd

After 2 months, the average score was

$$f(2) = 75 - 6 \ln(2 + 1)$$

Substitute 2 for t .

$$= 75 - 6 \ln 3$$

Simplify.

$$\approx 75 - 6(1.0986)$$

Use a calculator.

$$\approx 68.41.$$

Solution

Example 11(c) – *Solution*

cont'd

After 6 months, the average score was

$$f(6) = 75 - 6 \ln(6 + 1)$$

Substitute 6 for t .

$$= 75 - 6 \ln 7$$

Simplify.

$$\approx 75 - 6(1.9459)$$

Use a calculator.

$$\approx 63.32.$$

Solution