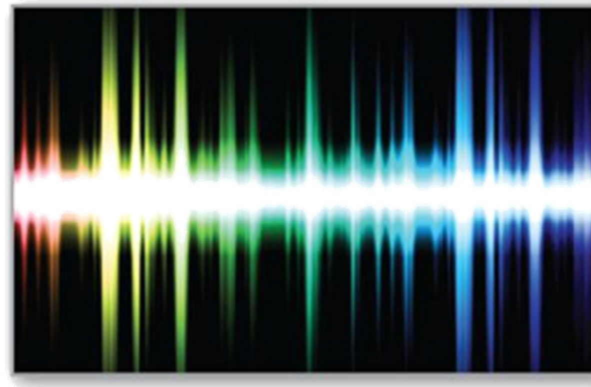
5 Exponential and Logarithmic Functions



5.1 Exponential Functions and Their Graphs

Objectives

- Recognize and evaluate exponential functions with base *a*.

- Graph exponential functions and use the One-to-One Property.

- Recognize, evaluate, and graph exponential functions with base *e*.

- Use exponential functions to model and solve real-life problems.

Exponential Functions

Exponential Functions

So far, this text has dealt mainly with **algebraic functions,** which include polynomial functions and rational functions.

In this chapter, you will study two types of nonalgebraic functions–*exponential functions* and *logarithmic functions*. These functions are examples of **transcendental functions.**

Definition of Exponential Function

The **exponential function *f* with base *a*** is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

Exponential Functions

The base $a = 1$ is excluded because it yields $f(x) = 1^x = 1$. This is a constant function, not an exponential function.

You have evaluated a^x for integer and rational values of x. For example, you know that $4^3 = 64$ and $4^{1/2} = 2$.

However, to evaluate 4^x for any real number x, you need to interpret forms with *irrational* exponents.

Exponential Functions

For the purposes of this text, it is sufficient to think of

$$a^{\sqrt{2}} \qquad \text{(where } \sqrt{2} \approx 1.41421356\text{)}$$

as the number that has the successively closer approximations

$$a^{1.4}, a^{1.41}, a^{1.414}, a^{1.4142}, a^{1.41421}, \ldots .$$

Example 1 – *Evaluating Exponential Functions*

Use a calculator to evaluate each function at the indicated value of *x*.

Function	**Value**
a. $f(x) = 2^x$	$x = -3.1$
b. $f(x) = 2^{-x}$	$x = \pi$
c. $f(x) = 0.6^x$	$x = \frac{3}{2}$

Example 1 – *Solution*

Function Value	Graphing Calculator Keystrokes	Display
a. $f(-3.1) = 2^{-3.1}$	2 [^] [(−)] 3.1 [ENTER]	0.1166291
b. $f(\pi) = 2^{-\pi}$	2 [^] [(−)] π [ENTER]	0.1133147
c. $f\left(\frac{3}{2}\right) = 0.6^{3/2}$	.6 [^] [(] 3 [÷] 2 [)] [ENTER]	0.4647580

Graphs of Exponential Functions

Example 2 – *Graphs of* $y = a^x$

In the same coordinate plane, sketch the graph of each function.

a. $f(x) = 2^x$

b. $g(x) = 4^x$

Example 2 – *Solution*

The following table lists some values for each function, and Figure 5.1 shows the graphs of the two functions.

x	-3	-2	-1	0	1	2
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
4^x	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16

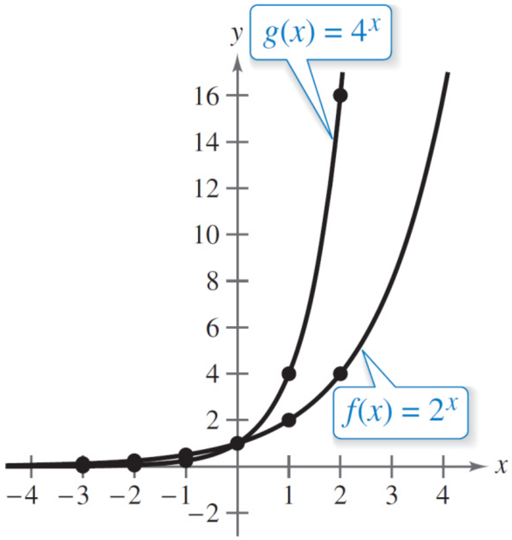


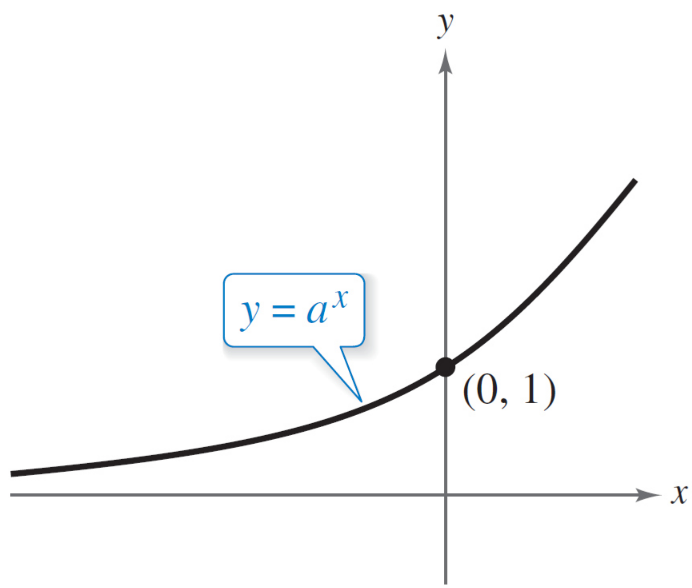
Figure 5.1

Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

Graphs of Exponential Functions

The following summarizes the basic characteristics of exponential functions $y = a^x$ and $y = a^{-x}$.

Graph of $y = a^x$, $a > 1$

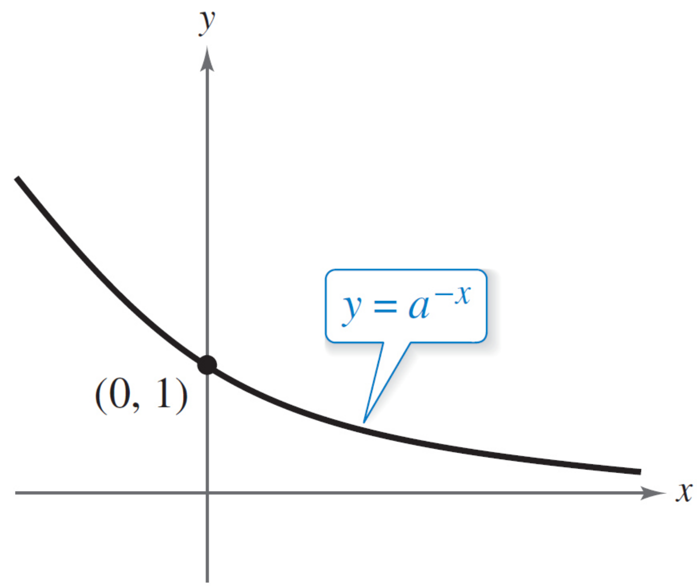


- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-intercept: (0, 1)
- Increasing
- x-axis is a horizontal asymptote ($a^x \to 0$ as $x \to -\infty$).
- Continuous

Graphs of Exponential Functions

Graph of $y = a^{-x}$, $a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-intercept: (0, 1)
- Decreasing
- x-axis is a horizontal asymptote ($a^{-x} \to 0$ as $x \to \infty$).
- Continuous



Notice that the graph of an exponential function is always increasing or always decreasing.

Graphs of Exponential Functions

As a result, the graphs pass the Horizontal Line Test, and therefore the functions are one-to-one functions.

You can use the following **One-to-One Property** to solve simple exponential equations.

For $a > 0$ and $a \neq 1$, $a^x = a^y$ if and only if $x = y$.

One-to-One Property

The Natural Base *e*

The Natural Base *e*

In many applications, the most convenient choice for a base is the irrational number

$e \approx 2.718281828 . . ..$

This number is called the **natural base.**

The function given by $f(x) = e^x$ is called the **natural exponential function.** Figure 5.7 shows its graph.

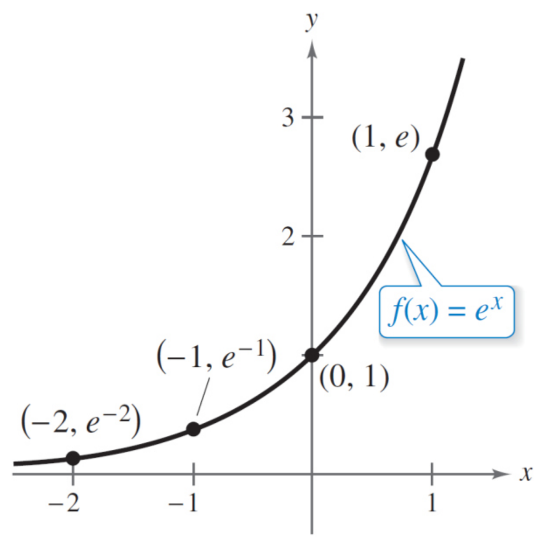


Figure 5.7

The Natural Base *e*

Be sure you see that for the exponential function $f(x) = e^x$, e is the constant 2.718281828 . . . , whereas x is the variable.

Example 6 – *Evaluating the Natural Exponential Function*

Use a calculator to evaluate the function $f(x) = e^x$ at each value of x.

a. $x = -2$

b. $x = -1$

c. $x = 0.25$

d. $x = -0.3$

Example 6 – *Solution*

Function Value	Graphing Calculator Keystrokes	Display
a. $f(-2) = e^{-2}$	e^x (−) 2 ENTER	0.1353353
b. $f(-1) = e^{-1}$	e^x (−) 1 ENTER	0.3678794
c. $f(0.25) = e^{0.25}$	e^x 0.25 ENTER	1.2840254
d. $f(-0.3) = e^{-0.3}$	e^x (−) 0.3 ENTER	0.7408182

Applications

Applications

One of the most familiar examples of exponential growth is an investment earning *continuously compounded interest*.

We know that the formula for *interest compounded n times per year* is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

In this formula, A is the balance in the account, P is the principal (or original deposit), r is the annual interest rate (in decimal form), n is the number of compoundings per year, and t is the time in years.

Applications

Using exponential functions, you can now *develop* this formula and show how it leads to continuous compounding.

Suppose you invest a principal P at an annual interest rate r, compounded once per year. If the interest is added to the principal at the end of the year, the new balance P_1 is

$$P_1 = P + Pr$$

$$= P(1 + r).$$

Applications

This pattern of multiplying the previous principal by $1 + r$ repeats each successive year, as follows.

Year	**Balance After Each Compounding**
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
3	$P_3 = P_2(1 + r) = P(1 + r)^2(1 + r) = P(1 + r)^3$
⋮	⋮
t	$P_t = P(1 + r)^t$

Applications

To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let *n* be the number of compoundings per year and let *t* be the number of years.

Then the rate per compounding is *r*/*n*, and the account balance after *t* years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Amount (balance) with *n* compoundings per year

When you let the number of compoundings *n* increase without bound, the process approaches what is called **continuous compounding.**

Applications

In the formula for n compoundings per year, let $m = n/r$. This produces

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Amount with n compoundings per year

$$= P\left(1 + \frac{r}{mr}\right)^{mrt}$$ Substitute mr for n.

$$= P\left(1 + \frac{1}{m}\right)^{mrt}$$ Simplify.

$$= P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$ Property of exponents

Applications

As m increases without bound (that is, as $m \to \infty$), the table at the right shows that $[1 + (1/m)]^m \to e$.

From this, you can conclude that the formula for continuous compounding is

$A = Pe^{rt}$. Substitute e for $(1 + 1/m)^m$.

m	$\left(1 + \frac{1}{m}\right)^m$
1	2
10	2.59374246
100	2.704813829
1,000	2.716923932
10,000	2.718145927
100,000	2.718268237
1,000,000	2.718280469
10,000,000	2.718281693
↓	↓
∞	e

Applications

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. For continuous compounding: $A = Pe^{rt}$

Example 8 – *Compound Interest*

You invest $12,000 at an annual rate of 3%. Find the balance after 5 years when the interest is compounded

a. quarterly.

b. monthly.

c. continuously.

Example 8(a) – *Solution*

For quarterly compounding, you have $n = 4$. So, in 5 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ Formula for compound interest

$$= 12{,}000\left(1 + \frac{0.03}{4}\right)^{4(5)}$$ Substitute for P, r, n, and t.

$$\approx \$13{,}934.21.$$ Use a calculator.

Example 8(b) – *Solution*

cont'd

For monthly compounding, you have $n = 12$. So, in 5 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Formula for compound interest

$$= 12{,}000\left(1 + \frac{0.03}{12}\right)^{12(5)}$$

Substitute for *P*, *r*, *n*, and *t*.

$$\approx \$13{,}939.40.$$

Use a calculator.

Example 8(c) – *Solution*

cont'd

For continuous compounding, the balance is

$A = Pe^{rt}$ Formula for continuous compounding

$= 12{,}000e^{0.03(5)}$ Substitute for P, r, and t.

$\approx \$13{,}942.01.$ Use a calculator.