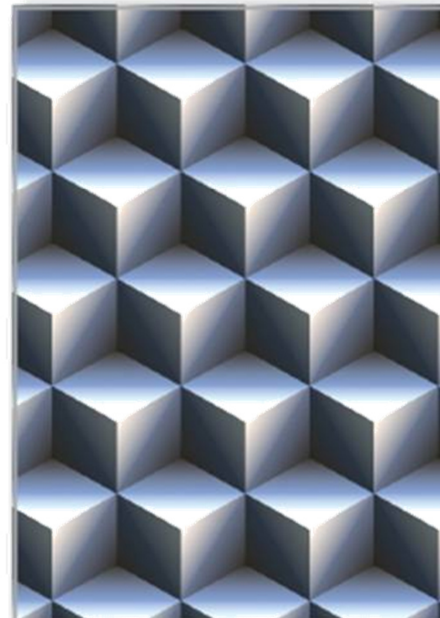
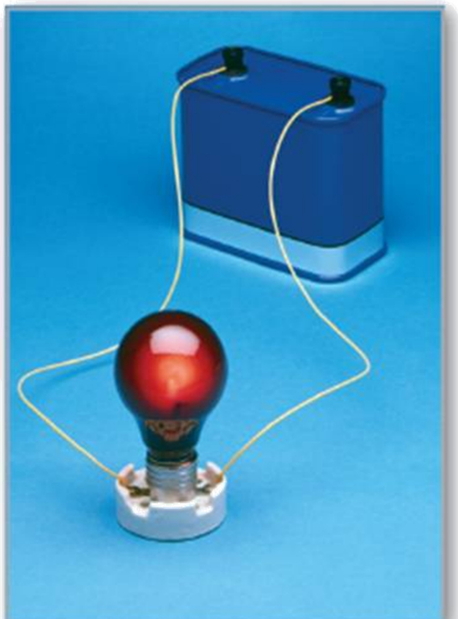
# 4 Complex Numbers





# 4.4 DeMoivre's Theorem

# Objectives

- Use DeMoivre's Theorem to find powers of complex numbers.
- Find *n*th roots of complex numbers.

# Powers of Complex Numbers

# Powers of Complex Numbers

The trigonometric form of a complex number is used to raise a complex number to a power.

To accomplish this, consider repeated use of the multiplication rule.

$$z = r(\cos \theta + i \sin \theta)$$

$$z^2 = r(\cos \theta + i \sin \theta) r(\cos \theta + i \sin \theta)$$

$$= r^2(\cos 2\theta + i \sin 2\theta)$$

$$z^3 = r^2(\cos 2\theta + i \sin 2\theta) r(\cos \theta + i \sin \theta)$$

$$= r^3(\cos 3\theta + i \sin 3\theta)$$

# Powers of Complex Numbers

$$z^4 = r^4(\cos 4\theta + i \sin 4\theta)$$

$$z^5 = r^5(\cos 5\theta + i \sin 5\theta)$$

$$\vdots$$

This pattern leads to **DeMoivre's Theorem**, which is named after the French mathematician Abraham DeMoivre.

**DeMoivre's Theorem**

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and $n$ is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

# Example 1 – *Finding a Power of a Complex Number*

Use DeMoivre's Theorem to find $(-1 + \sqrt{3i})^{12}$.

Solution:

The absolute value of $z = -1 + \sqrt{3}i$ is

$$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2$$

and the argument $\theta$ is determined from $\tan \theta = \sqrt{3}/(-1)$.

Because $z = -1 + \sqrt{3}i$ lies in Quadrant II,

$$\theta = \pi + \arctan \frac{\sqrt{3}}{-1} = \pi + \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}.$$

# Example 1 – *Solution*

cont'd

So, the trigonometric form is

$$z = -1 + \sqrt{3}\,i \;\; = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Then, by DeMoivre's Theorem, you have

$$\left(-1 + \sqrt{3}\,i\right)^{12} = \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{12}$$

$$= 2^{12}\left[\cos\frac{12(2\pi)}{3} + i\sin\frac{12(2\pi)}{3}\right]$$

$$= 4096(\cos 8\pi + i\sin 8\pi)$$

$$= 4096.$$

# Roots of Complex Numbers

# Roots of Complex Numbers

You know that a consequence of the Fundamental Theorem of Algebra is that a polynomial equation of degree $n$ has $n$ solutions in the complex number system.

So, the equation $x^6 = 1$ has six solutions, and in this particular case you can find the six solutions by factoring and using the Quadratic Formula.

$$x^6 - 1 = 0$$

$$(x^3 - 1)(x^3 + 1) = 0$$

$$(x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1) = 0$$

# Roots of Complex Numbers

Consequently, the solutions are

$$x = \pm 1, \qquad x = \frac{-1 \pm \sqrt{3}\,i}{2}, \qquad \text{and} \qquad x = \frac{1 \pm \sqrt{3}\,i}{2}.$$

Each of these numbers is a sixth root of 1. In general, an ***n*th root of a complex number** is defined as follows.

**Definition of an *n*th Root of a Complex Number**

The complex number $u = a + bi$ is an ***n*th root** of the complex number $z$ when

$$\begin{aligned} z &= u^n \\ &= (a + bi)^n. \end{aligned}$$

# Roots of Complex Numbers

To find a formula for an $n$th root of a complex number, let $u$ be an $n$th root of $z$, where

$$u = s(\cos \beta + i \sin \beta)$$

and

$$z = r(\cos \theta + i \sin \theta).$$

By DeMoivre's Theorem and the fact that $u^n = z$, you have

$$s^n(\cos n\beta + i \sin n\beta) = r(\cos \theta + i \sin \theta).$$

# Roots of Complex Numbers

Taking the absolute value of each side of this equation, it follows that $s^n = r$. Substituting back into the previous equation and dividing by $r$, you get

$$\cos n\beta + i \sin n\beta = \cos \theta + i \sin \theta.$$

So, it follows that

$$\cos n\beta = \cos \theta \quad \text{and} \quad \sin n\beta = \sin \theta.$$

Because both sine and cosine have a period of $2\pi$, these last two equations have solutions if and only if the angles differ by a multiple of $2\pi$.

# Roots of Complex Numbers

Consequently, there must exist an integer *k* such that

$$n\beta = \theta + 2\pi k$$

$$\beta = \frac{\theta + 2\pi k}{n}.$$

By substituting this value of $\beta$ into the trigonometric form of *u*, you get the result stated below.

**Finding *n*th Roots of a Complex Number**

For a positive integer $n$, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly $n$ distinct $n$th roots given by

$$z_k = \sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

where $k = 0, 1, 2, \ldots, n - 1$.

# Roots of Complex Numbers

When $k > n - 1$, the roots begin to repeat.

For instance, if $k = n$, the angle

$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi$$

is coterminal with $\theta/n$, which is also obtained when $k = 0$.

The formula for the $n$th roots of a complex number $z$ has a nice geometrical interpretation, as shown in Figure 4.6.

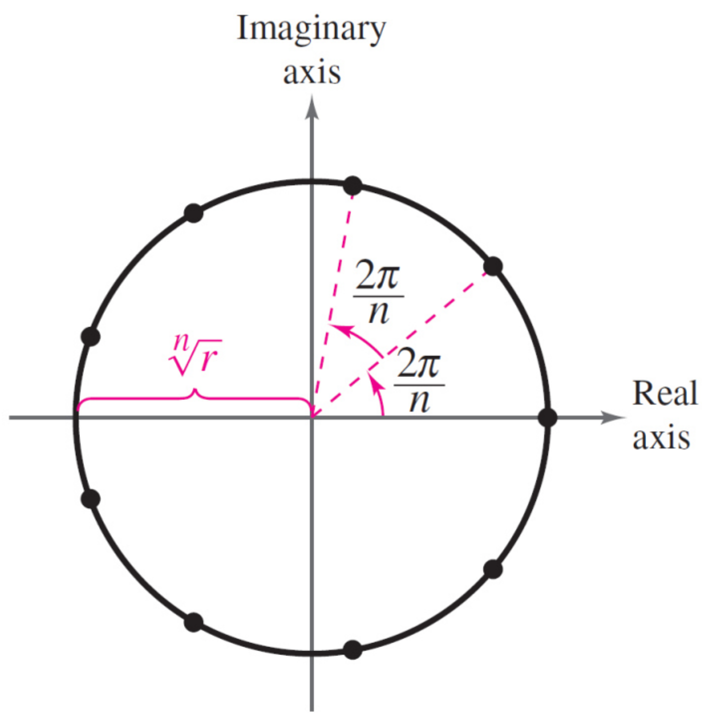



**Figure 4.6**

# Roots of Complex Numbers

Note that because the *n*th roots of *z* all have the same magnitude $\sqrt[n]{r}$, they all lie on a circle of radius $\sqrt[n]{r}$ with center at the origin.

Furthermore, because successive *n*th roots have arguments that differ by $2\pi/n$, the *n* roots are equally spaced around the circle.

You have already found the sixth roots of 1 by factoring and using the Quadratic Formula.

The *n* distinct *n*th roots of 1 are called the ***n*th roots of unity.**

## Example 3 – *Finding the nth Roots of a Complex Number*

Find the three cube roots of $z = -2 + 2i$.

Solution:

The absolute value of *z is*

$$r = | -2 + 2i | = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$$

and the argument $\theta$ is determined from

$$\tan \theta = \frac{b}{a} = \frac{2}{-2} = -1.$$

# Example 3 – *Solution*

cont'd

Because $z$ lies in Quadrant II, the trigonometric form of $z$ is

$$z = -2 + 2i$$

$$= \sqrt{8}\,(\cos 135° + i \sin 135°).$$ $\theta = \pi + \arctan(-1) = 3\pi/4 = 135°$

By the formula for $n$th roots, the cube roots have the form

$$z_k = \sqrt[6]{8}\left(\cos \frac{135° + 360°k}{3} + i \sin \frac{135° + 360°k}{3}\right).$$

# Example 3 – *Solution* cont'd

Finally, for $k = 0$, 1, and 2, you obtain the roots

$$z_0 = \sqrt[6]{8}\left(\cos\frac{135^\circ + 360^\circ(0)}{3} + i\sin\frac{135^\circ + 360^\circ(0)}{3}\right)$$

$$= \sqrt{2}(\cos 45^\circ + i\sin 45^\circ)$$

$$= 1 + i$$

$$z_1 = \sqrt[6]{8}\left(\cos\frac{135^\circ + 360^\circ(1)}{3} + i\sin\frac{135^\circ + 360^\circ(1)}{3}\right)$$

$$= \sqrt{2}(\cos 165^\circ + i\sin 165^\circ)$$

$$\approx -1.3660 + 0.3660i$$

# Example 3 – *Solution*

cont'd

$$z_2 = \sqrt[6]{8}\left(\cos\frac{135^\circ + 360^\circ(2)}{3} + i\sin\frac{135^\circ + 360^\circ(2)}{3}\right)$$

$$= \sqrt{2}(\cos 285^\circ + i\sin 285^\circ)$$

$$\approx 0.3660 - 1.3660i.$$

See Figure 4.8.

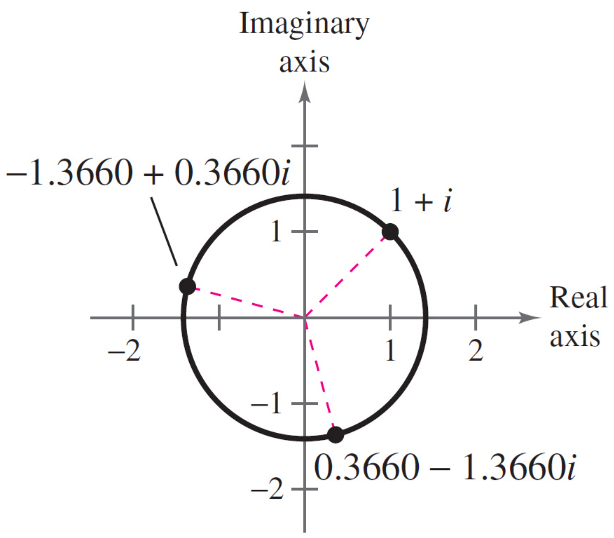



Figure 4.8