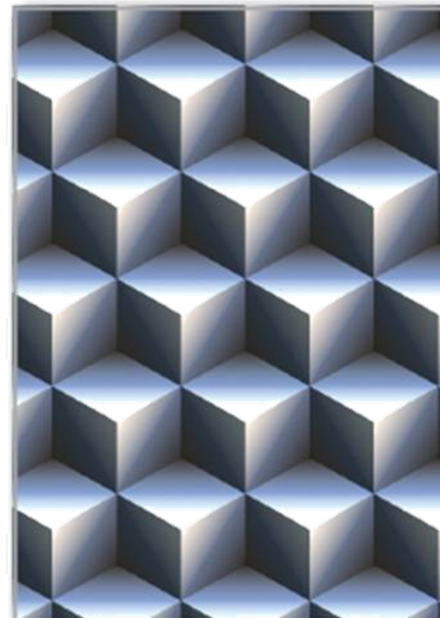
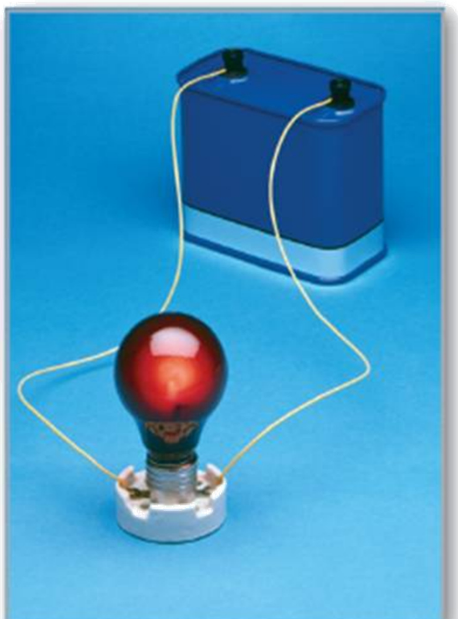
4 Complex Numbers





4.3 Trigonometric Form of a Complex Number

Objectives

- Plot complex numbers in the complex plane and find absolute values of complex numbers.

- Write the trigonometric forms of complex numbers.

- Multiply and divide complex numbers written in trigonometric form.

The Complex Plane

The Complex Plane

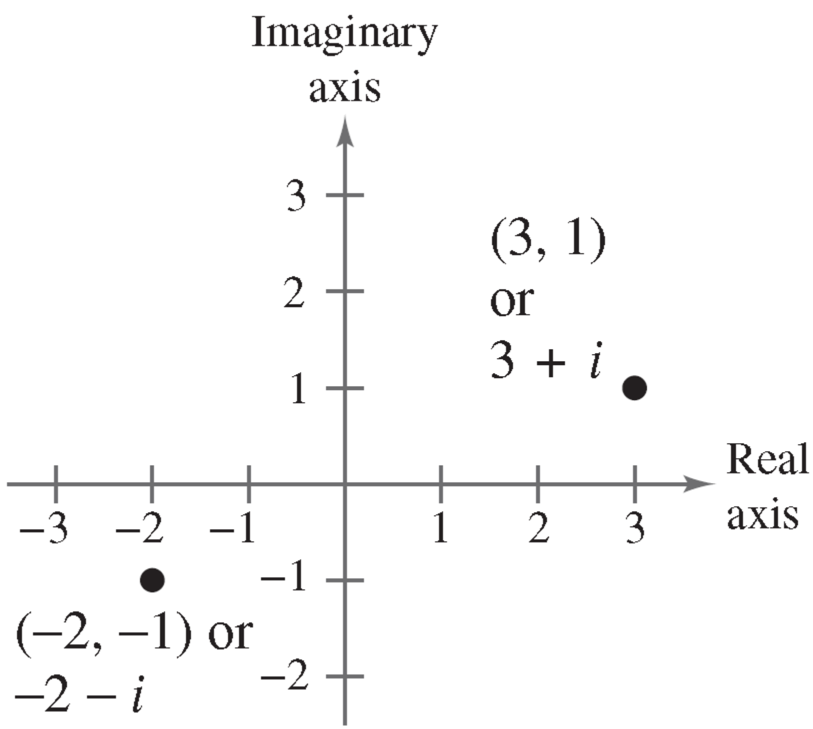
Just as real numbers can be represented by points on the real number line, you can represent a complex number

$$z = a + bi$$

as the point (a, b) in a coordinate plane (the **complex plane**).

The Complex Plane

The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**, as shown below.



The Complex Plane

The **absolute value** of the complex number $a + bi$ is defined as the distance between the origin (0, 0) and the point (a, b).

Definition of the Absolute Value of a Complex Number

The **absolute value** of the complex number $z = a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}.$$

When the complex number $a + bi$ is a real number (that is, if $b = 0$), this definition agrees with that given for the absolute value of a real number

$$|a + 0i| = \sqrt{a^2 + 0^2} = |a|.$$

Example 1 – *Finding the Absolute Value of a Complex Number*

Plot $z = -2 + 5i$ and find its absolute value.

Solution:

The number is plotted in Figure 4.2.

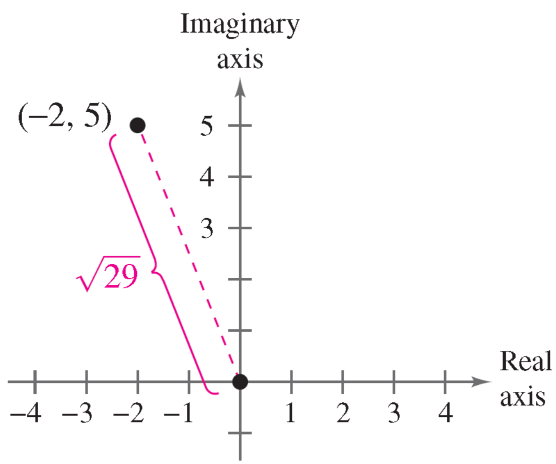


Figure 4.2

It has an absolute value of

$$|z| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}.$$

Trigonometric Form of a Complex Number

Trigonometric Form of a Complex Number

You have learned how to add, subtract, multiply, and divide complex numbers.

To work effectively with *powers* and *roots* of complex numbers, it is helpful to write complex numbers in trigonometric form.

In Figure 4.3, consider the nonzero complex number $a + bi$.

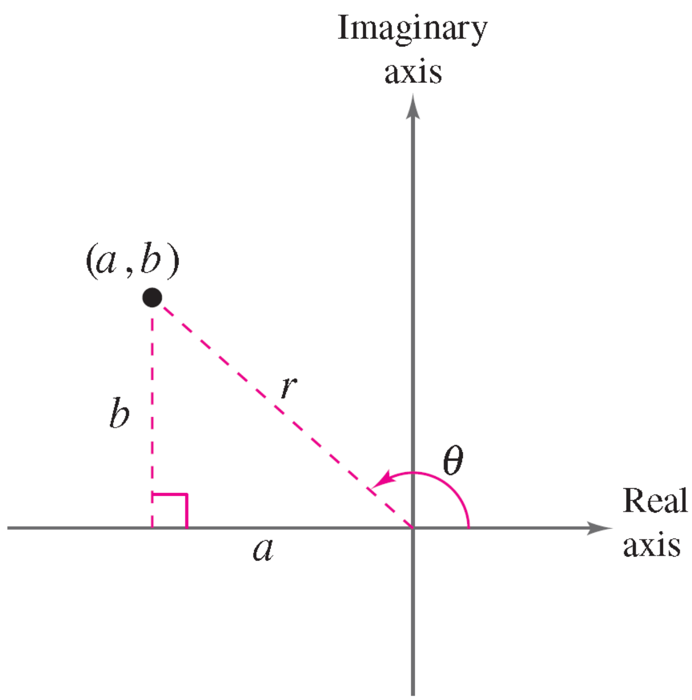


Figure 4.3

Trigonometric Form of a Complex Number

By letting θ be the angle from the positive real axis (measured counterclockwise) to the line segment connecting the origin and the point (a, b), you can write

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

where $r = \sqrt{a^2 + b^2}$.

Consequently, you have

$$a + bi = (r\cos\theta) + (r\sin\theta)i$$

from which you can obtain the **trigonometric form of a complex number**.

Trigonometric Form of a Complex Number

Trigonometric Form of a Complex Number

The **trigonometric form** of the complex number $z = a + bi$ is

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is called an **argument** of z.

The trigonometric form of a complex number is also called the *polar form*. Because there are infinitely many choices for θ, the trigonometric form of a complex number is not unique.

Normally, θ is restricted to the interval $0 \le \theta < 2\pi$, although on occasion it is convenient to use $\theta < 0$.

Example 2 – *Trigonometric Form of a Complex Number*

Write the complex number $z = -2 - 2\sqrt{3}\,i$ in trigonometric form.

Solution:

The absolute value of z is

$$r = \left|-2 - 2\sqrt{3}i\right| = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{16} = 4$$

and the argument angle θ is determined from

$$\tan\theta = \frac{b}{a} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}.$$

Example 2 – *Solution*

cont'd

Because $z = -2 - 2\sqrt{3}\,i$ lies in Quadrant III, as shown in Figure 4.4, $\theta = \pi + \arctan\sqrt{3} = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

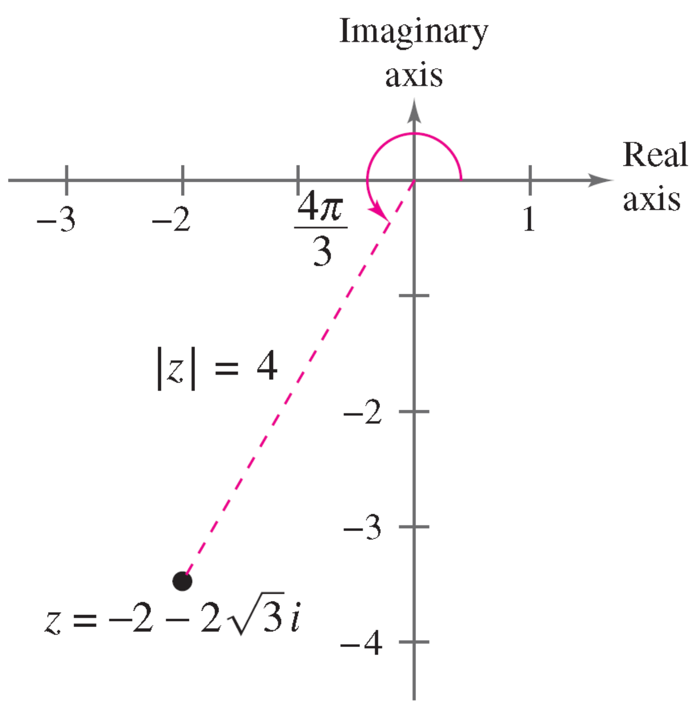


Figure 4.4

Example 2 – *Solution*

cont'd

So, the trigonometric form is

$$z = r(\cos\theta + i\sin\theta)$$

$$= 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right).$$

Multiplication and Division of Complex Numbers

Multiplication and Division of Complex Numbers

The trigonometric form adapts nicely to multiplication and division of complex numbers.

Suppose you are given two complex numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

The product of z_1 and z_2 is

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

Multiplication and Division of Complex Numbers

Using the sum and difference formulas for cosine and sine, you can rewrite this equation as

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].$$

This establishes the first part of the following rule.

Product and Quotient of Two Complex Numbers

Let

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

be complex numbers.

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \qquad \text{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)], \quad z_2 \neq 0 \qquad \text{Quotient}$$

Multiplication and Division of Complex Numbers

Note that this rule says that to *multiply* two complex numbers you multiply moduli and add arguments, whereas to *divide* two complex numbers you divide moduli and subtract arguments.

Example 6 – *Multiplying Complex Numbers*

Find the product z_1z_2 of the complex numbers.

$z_1 = 2(\cos 120° + i \sin 120°)$

$z_2 = 8(\cos 330° + i \sin 330°)$

Solution:

$$z_1z_2 = 2(\cos 120° + i \sin 120°) \cdot 8(\cos 330° + i \sin 330°)$$

$$= 16[\cos(120° + 330°) + i \sin(120° + 330°)]$$

Multiply moduli and add arguments.

$$= 16(\cos 450° + i \sin 450°)$$

Example 6 – *Solution*

cont'd

$= 16(\cos 90° + i \sin 90°)$ 450° and 90° are coterminal.

$= 16[0 + i(1)]$

$= 16i$

Example 7 – *Dividing Complex Numbers*

Find the quotient z_1/z_2 of the complex numbers.

$$z_1 = 24\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

$$z_2 = 8\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

Solution:

$$\frac{z_1}{z_2} = \frac{24[\cos(5\pi/3) + i\sin(5\pi/3)]}{8[\cos(5\pi/12) + i\sin(5\pi/12)]}$$

$$= 3\left[\cos\left(\frac{5\pi}{3} - \frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{3} - \frac{5\pi}{12}\right)\right]$$

Divide moduli and subtract arguments.

Example 7 – *Solution*

cont'd

$$= 3\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

$$= 3\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$

$$= -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$$