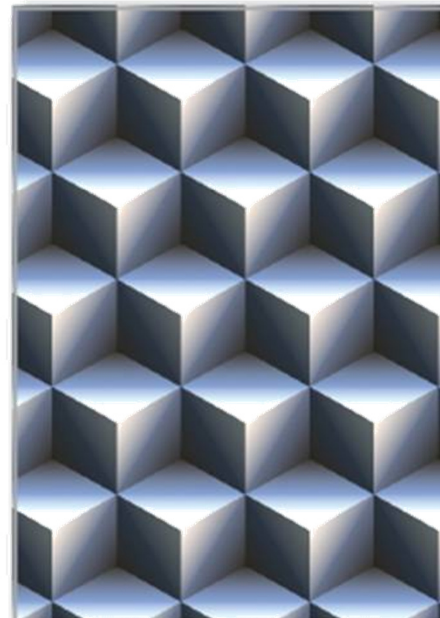
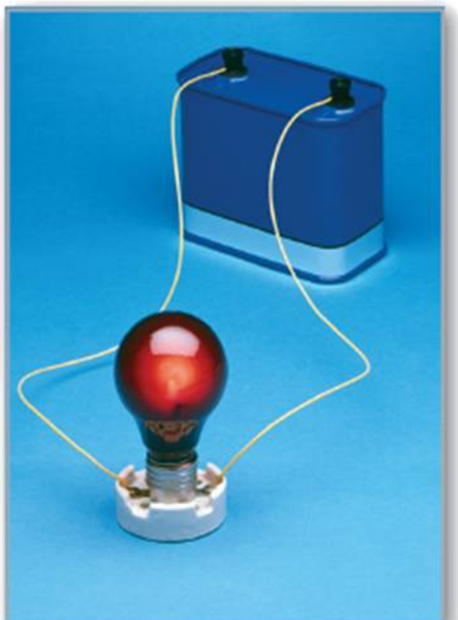
4 Complex Numbers





4.2 Complex Solutions of Equations

Objectives

- Determine the numbers of solutions of polynomial equations.
- Find solutions of polynomial equations.
- Find zeros of polynomial functions and find polynomial functions given the zeros of the functions.

The Number of Solutions of a Polynomial Equation

The Number of Solutions of a Polynomial Equation

The Fundamental Theorem of Algebra implies that a polynomial equation of degree n has precisely n solutions in the complex number system.

These solutions can be real or complex and may be repeated. The Fundamental Theorem of Algebra and the Linear Factorization Theorem are listed below.

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

The Number of Solutions of a Polynomial Equation

Note that finding zeros of a polynomial function *f* is equivalent to finding solutions of the polynomial equation $f(x) = 0$.

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n, where $n > 0$, then $f(x)$ has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Example 1 – *Solutions of Polynomial Equations*

a. The first-degree equation $x - 2 = 0$ has exactly *one* solution: $x = 2$.

b. The second-degree equation

$$x^2 - 6x + 9 = 0$$ Second-degree equation

$$(x - 3)(x - 3) = 0$$ Factor.

has exactly *two* solutions: $x = 3$ and $x = 3$. (This is called a *repeated solution*.)

Example 1 – *Solutions of Polynomial Equations* cont'd

c. The fourth-degree equation

$$x^4 - 1 = 0 \quad \text{Fourth-degree equation}$$

$$(x - 1)(x + 1)(x - i)(x + i) = 0 \quad \text{Factor.}$$

has exactly *four* solutions: $x = 1$, $x = -1$, $x = i$, and $x = -i$.

The Number of Solutions of a Polynomial Equation

You can use a graph to check the number of real solutions of an equation. As shown in Figure 4.1, the graph of $f(x) = x^4 - 1$ has two x-intercepts, which implies that the equation has two real solutions.

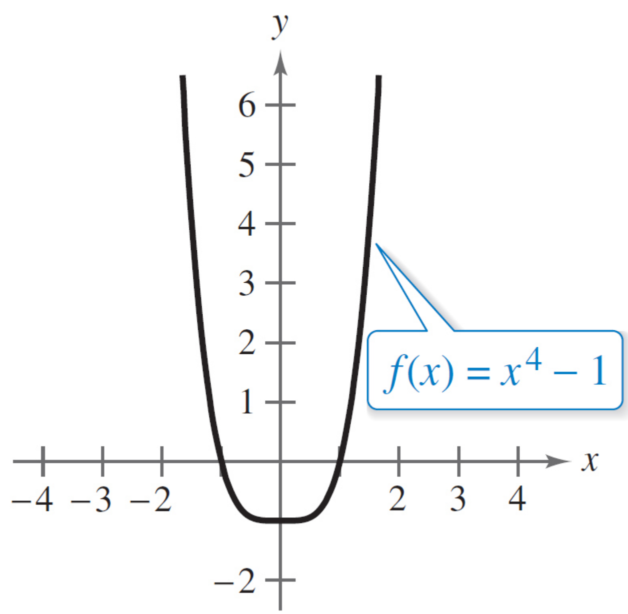


Figure 4.1

The Number of Solutions of a Polynomial Equation

Every second-degree equation, $ax^2 + bx + c = 0$, has precisely two solutions given by the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression inside the radical, $b^2 - 4ac$, is called the **discriminant,** and can be used to determine whether the solutions are real, repeated, or complex.

The Number of Solutions of a Polynomial Equation

1. If $b^2 - 4ac < 0$, then the equation has two complex solutions.

2. If $b^2 - 4ac = 0$, then the equation has one repeated real solution.

3. If $b^2 - 4ac > 0$, then the equation has two distinct real solutions.

Finding Solutions of Polynomial Equations

Example 3 – *Solving a Quadratic Equation*

Solve $x^2 + 2x + 2 = 0$. Write complex solutions in standard form.

Solution:

Using $a = 1$, $b = 2$, and $c = 2$, you can apply the Quadratic Formula as follows.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula

$$= \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$$

Substitute 1 for *a*, 2 for *b*, and 2 for *c*.

Example 3 – *Solution*

cont'd

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

Simplify.

$$= \frac{-2 \pm 2i}{2}$$

Simplify.

$$= -1 \pm i$$

Write in standard form.

Finding Solutions of Polynomial Equations

In Example 3, the two complex solutions are *conjugates*. That is, they are of the form $a \pm bi$. This is not a coincidence, as indicated by the following theorem.

Complex Solutions Occur in Conjugate Pairs

If $a + bi$, $b \neq 0$, is a solution of a polynomial equation with real coefficients, then the conjugate $a - bi$ is also a solution of the equation.

Be sure you see that this result is true only when the polynomial has *real* coefficients. For instance, the result applies to the equation $x^2 + 1 = 0$, but not to the equation $x - i = 0$.

Finding Zeros of Polynomial Functions

Finding Zeros of Polynomial Functions

The problem of finding the *zeros* of a polynomial function is essentially the same problem as finding the solutions of a polynomial equation.

For instance, the zeros of the polynomial function

$$f(x) = 3x^2 - 4x + 5$$

are simply the solutions of the polynomial equation

$$3x^2 - 4x + 5 = 0.$$

Example 5 – *Finding the Zeros of a Polynomial Function*

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that is $1 + 3i$ is a zero of f.

Solution:

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f.

This means that both $[x - (1 + 3i)]$ and $[x - (1 - 3i)]$ are factors of $f(x)$.

Example 5 – *Solution*

cont'd

Multiplying these two factors produces

$$[x - (1 + 3i)][x - (1 - 3i)] = [(x - 1) - 3i][(x - 1) + 3i]$$

$$= (x - 1)^2 - 9i^2$$

$$= x^2 - 2x + 10.$$

Example 5 – *Solution*

cont'd

Using long division, you can divide $x^2 - 2x + 10$ into $f(x)$ to obtain the following.

$$
\begin{array}{r}
x^2 - x - 6 \\
x^2 - 2x + 10 \overline{\smash{)}\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\
\underline{x^4 - 2x^3 + 10x^2} \qquad\qquad \\
-x^3 - 4x^2 + 2x \qquad \\
\underline{-x^3 + 2x^2 - 10x} \qquad \\
-6x^2 + 12x - 60 \\
\underline{-6x^2 + 12x - 60} \\
0
\end{array}
$$

Example 5 – *Solution*

cont'd

So, you have

$$f(x) = (x^2 - 2x + 10)(x^2 - x - 6)$$

$$= (x^2 - 2x + 10)(x - 3)(x + 2)$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

Example 6 – *Finding a Polynomial Function with Given Zeros*

Find a fourth-degree polynomial function with real coefficients that has –1, –1, and $3i$ as zeros.

Solution:

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero.

So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

Example 6 – *Solution*

cont'd

For simplicity, let $a = 1$ to obtain

$$f(x) = (x^2 + 2x + 1)(x^2 + 9)$$

$$= x^4 + 2x^3 + 10x^2 + 18x + 9.$$