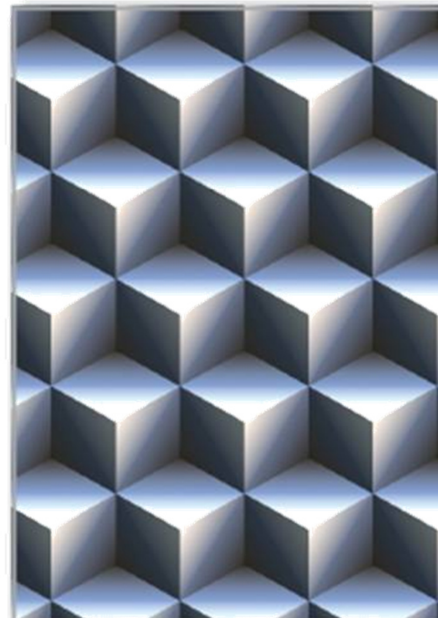
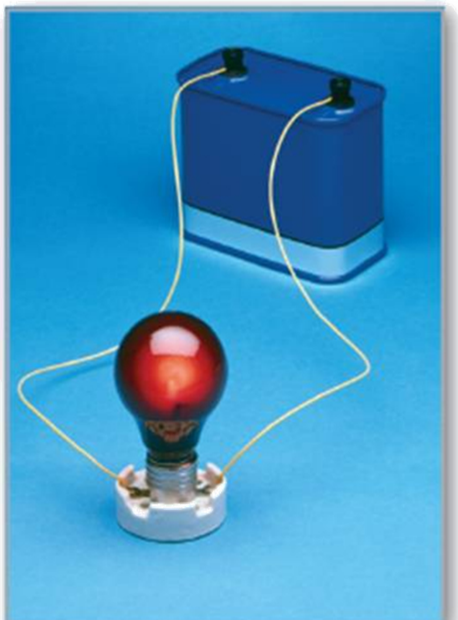
4 Complex Numbers





4.1 Complex Numbers

Objectives

- Use the imaginary unit *i* to write complex numbers.
- Add, subtract, and multiply complex numbers.
- Use complex conjugates to write the quotient of two complex numbers in standard form.
- Find complex solutions of quadratic equations.

The Imaginary Unit *i*

The Imaginary Unit *i*

Some quadratic equations have no real solutions. For instance, the quadratic equation

$$x^2 + 1 = 0$$

has no real solution because there is no real number x that can be squared to produce –1.

To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit *i***, defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$.

The Imaginary Unit i

By adding real numbers to real multiples of this imaginary unit, the set of **complex numbers** is obtained.

Each complex number can be written in the **standard form** ***a* + *bi***.

For instance, the standard form of the complex number $-5 + \sqrt{-9}$ is $-5 + 3i$ because

$$-5 + \sqrt{-9} = -5 + \sqrt{3^2(-1)} = -5 + 3\sqrt{-1} = -5 + 3i.$$

The Imaginary Unit *i*

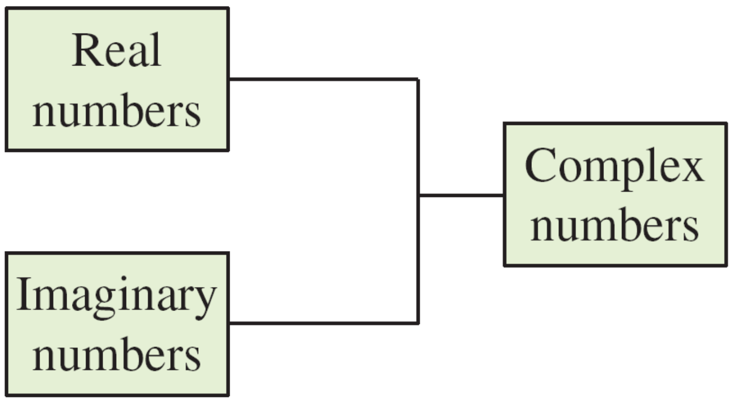
Definition of a Complex Number

Let a and b be real numbers. The number $a + bi$ is called a **complex number,** and it is said to be written in **standard form.** The real number a is called the **real part** and the real number b is called the **imaginary part** of the complex number.

When $b = 0$, the number $a + bi$ is a real number. When $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The Imaginary Unit *i*

The set of real numbers is a subset of the set of complex numbers, as shown below.



This is true because every real number a can be written as a complex number using $b = 0$.

That is, for every real number a, you can write $a = a + 0i$.

The Imaginary Unit i

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

if and only if $a = c$ and $b = d$.

Operations with Complex Numbers

Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

For two complex numbers $a + bi$ and $c + di$ written in standard form, the sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

Operations with Complex Numbers

The **additive identity** in the complex number system is zero (the same as in the real number system).

Furthermore, the **additive inverse** of the complex number $a + bi$ is

$$-(a + bi) = -a - bi. \qquad \text{Additive inverse}$$

So, you have

$$(a + bi) + (-a - bi) = 0 + 0i = 0.$$

Example 1 – *Adding and Subtracting Complex Numbers*

a. $(4 + 7i) + (1 - 6i) = 4 + 7i + 1 - 6i$ Remove parentheses.

$= (4 + 1) + (7 - 6)i$ Group like terms.

$= 5 + i$ Write in standard form.

b. $(1 + 2i) + (3 - 2i) = 1 + 2i + 3 - 2i$ Remove parentheses.

$= (1 + 3) + (2 - 2)i$ Group like terms.

$= 4 + 0i$ Simplify.

$= 4$ Write in standard form.

Example 1 – *Adding and Subtracting Complex Numbers* cont'd

c. $3i - (-2 + 3i) - (2 + 5i) = 3i + 2 - 3i - 2 - 5i$

$= (2 - 2) + (3 - 3 - 5)i$

$= 0 - 5i$

$= -5i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= (3 + 4 - 7) + (2 - 1 - 1)i$

$= 0 + 0i$

$= 0$

Operations with Complex Numbers

Many of the properties of real numbers are valid for complex numbers as well.

Here are some examples.

Associative Properties of Addition and Multiplication

Commutative Properties of Addition and Multiplication

Distributive Property of Multiplication Over Addition

Operations with Complex Numbers

Notice below how these properties are used when two complex numbers are multiplied.

$(a + bi)(c + di) = a(c + di) + bi(c + di)$	Distributive Property
$= ac + (ad)i + (bc)i + (bd)i^2$	Distributive Property
$= ac + (ad)i + (bc)i + (bd)(-1)$	$i^2 = -1$
$= ac - bd + (ad)i + (bc)i$	Commutative Property
$= (ac - bd) + (ad + bc)i$	Associative Property

Rather than trying to memorize this multiplication rule, you should simply remember how to use the Distributive Property to multiply two complex numbers.

Example 2 – *Multiplying Complex Numbers*

a. $4(-2 + 3i) = 4(-2) + 4(3i)$ — Distributive Property

$= -8 + 12i$ — Simplify.

b. $(2 - i)(4 + 3i) = 2(4 + 3i) - i(4 + 3i)$ — Distributive Property

$= 8 + 6i - 4i - 3i^2$ — Distributive Property

$= 8 + 6i - 4i - 3(-1)$ — $i^2 = -1$

$= (8 + 3) + (6 - 4)i$ — Group like terms.

$= 11 + 2i$ — Write in standard form.

Example 2 – *Multiplying Complex Numbers* cont'd

c. $(3 + 2i)(3 - 2i) = 3(3 - 2i) + 2i(3 - 2i)$ Distributive Property

$= 9 - 6i + 6i - 4i^2$ Distributive Property

$= 9 - 6i + 6i - 4(-1)$ $i^2 = -1$

$= 9 + 4$ Simplify.

$= 13$ Write in standard form.

Example 2 – *Multiplying Complex Numbers* cont'd

d. $(3 + 2i)^2 = (3 + 2i)(3 + 2i)$ — Square of a binomial

$= 3(3 + 2i) + 2i(3 + 2i)$ — Distributive Property

$= 9 + 6i + 6i + 4i^2$ — Distributive Property

$= 9 + 6i + 6i + 4(-1)$ — $i^2 = -1$

$= 9 + 12i - 4$ — Simplify.

$= 5 + 12i$ — Write in standard form.

Complex Conjugates

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number.

This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates**.

$$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2$$

$$= a^2 - b^2(-1)$$

$$= a^2 + b^2$$

Complex Conjugates

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right)$$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

Example 4 – *Quotient of Complex Numbers in Standard Form*

$$\frac{2+3i}{4-2i} = \frac{2+3i}{4-2i}\left(\frac{4+2i}{4+2i}\right)$$ Multiply numerator and denominator by complex conjugate of denominator.

$$= \frac{8+4i+12i+6i^2}{16-4i^2}$$ Expand.

$$= \frac{8-6+16i}{16+4}$$ $i^2 = -1$

$$= \frac{2+16i}{20}$$ Simplify.

$$= \frac{1}{10} + \frac{4}{5}i$$ Write in standard form.

Complex Solutions of Quadratic Equations

Complex Solutions of Quadratic Equations

You can write a number such as $\sqrt{-3}$ in standard form by factoring out $i = \sqrt{-1}$.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is called the *principal square root* of –3.

Principal Square Root of a Negative Number

When a is a positive real number, the **principal square root** of $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}i.$$

Example 6 – *Complex Solutions of a Quadratic Equation*

Solve $3x^2 - 2x + 5 = 0$.

Solution:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$$ Quadratic Formula

$$= \frac{2 \pm \sqrt{-56}}{6}$$ Simplify.

$$= \frac{2 \pm 2\sqrt{14}\,i}{6}$$ Write $\sqrt{-56}$ in standard form.

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i$$ Write in standard form.