3 Additional Topics in Trigonometry

3.4 Vectors and Dot Products

Objectives

- Find the dot product of two vectors and use the properties of the dot product.
- Find the angle between two vectors and determine whether two vectors are orthogonal.
- Write a vector as the sum of two vector components.
- Use vectors to find the work done by a force.

The Dot Product of Two Vectors

The Dot Product of Two Vectors

In this section, you will study a third vector operation, the **dot product.** This operation yields a scalar, rather than a vector.

Definition of the Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

The Dot Product of Two Vectors

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

Example 1 – *Finding Dot Products*

a. $\langle 4, 5\rangle \cdot \langle 2, 3\rangle = 4(2) + 5(3)$

$= 8 + 15$

$= 23$

b. $\langle 2, -1\rangle \cdot \langle 1, 2\rangle = 2(1) + (-1)(2)$

$= 2 - 2$

$= 0$

Example 1 – *Finding Dot Products* cont'd

c. $\langle 0, 3\rangle \cdot \langle 4, -2\rangle = 0(4) + 3(-2)$

$$= 0 - 6$$

$$= -6$$

The Angle Between Two Vectors

The Angle Between Two Vectors

The **angle between two nonzero vectors** is the angle θ, $0 \leq \theta \leq \pi$, between their respective standard position vectors, as shown in Figure 3.27. This angle can be found using the dot product.

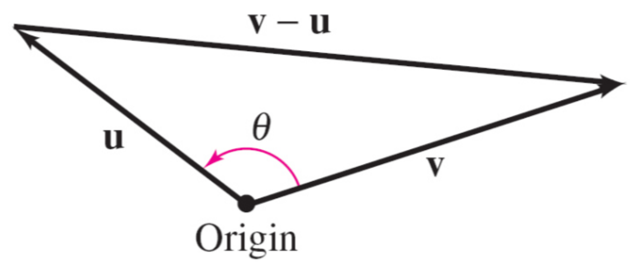


Figure 3.27

Angle Between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Example 3 – *Finding the Angle Between Two Vectors*

Find the angle θ between $\mathbf{u} = \langle 4, 3 \rangle$ and $\mathbf{v} = \langle 3, 5 \rangle$ (see Figure 3.28).

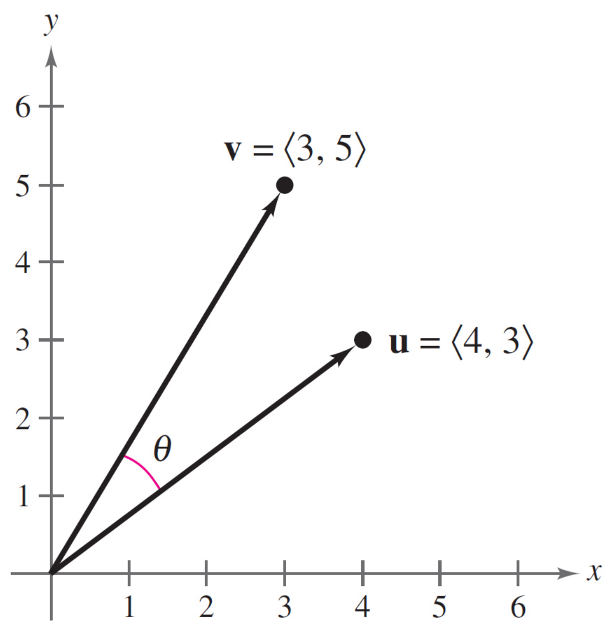


Figure 3.28

Example 3 – *Solution*

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$$

$$= \frac{\langle 4, 3\rangle \cdot \langle 3, 5\rangle}{\|\langle 4, 3\rangle\| \, \|\langle 3, 5\rangle\|}$$

$$= \frac{4(3) + 3(5)}{\sqrt{4^2 + 3^2}\sqrt{3^2 + 5^2}}$$

$$= \frac{27}{5\sqrt{34}}$$

Example 3 – *Solution*

cont'd

This implies that the angle between the two vectors is

$$\theta = \cos^{-1} \frac{27}{5\sqrt{34}}$$

$$\approx 0.3869 \text{ radian.}$$

Use a calculator.

The Angle Between Two Vectors

Rewriting the expression for the angle between two vectors in the form

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta$$

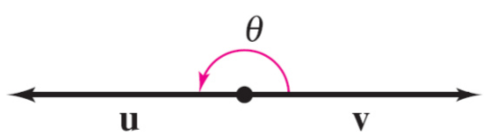
Alternative form of dot product

produces an alternative way to calculate the dot product.

From this form, you can see that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u} \cdot \mathbf{v}$ and $\cos \theta$ will always have the same sign.

The Angle Between Two Vectors

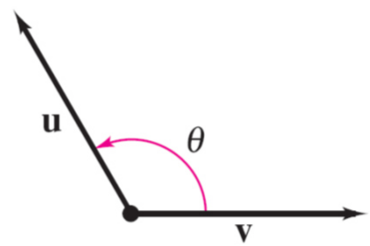
The five possible orientations of two vectors are shown below.



$\theta = \pi$

$\cos\theta = -1$

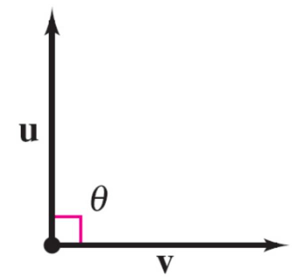
Opposite Direction



$\frac{\pi}{2} < \theta < \pi$

$-1 < \cos\theta < 0$

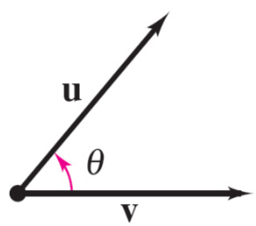
Obtuse Angle



$\theta = \frac{\pi}{2}$

$\cos\theta = 0$

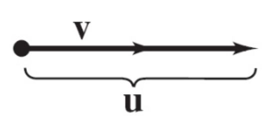
$90°$ Angle



$0 < \theta < \frac{\pi}{2}$

$0 < \cos\theta < 1$

Acute Angle



$\theta = 0$

$\cos\theta = 1$

Same Direction

The Angle Between Two Vectors

Definition of Orthogonal Vectors

The vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

The terms *orthogonal* and *perpendicular* mean essentially the same thing—meeting at right angles.

Note that the zero vector is orthogonal to every vector $\mathbf{u}$, because $\mathbf{0} \cdot \mathbf{u} = 0$.

Finding Vector Components

Finding Vector Components

You have already seen applications in which two vectors are added to produce a resultant vector. Many applications in physics and engineering pose the reverse problem—decomposing a given vector into the sum of two **vector components**.

Consider a boat on an inclined ramp, as shown in Figure 3.29.

The force **F** due to gravity pulls the boat *down* the ramp and *against* the ramp.

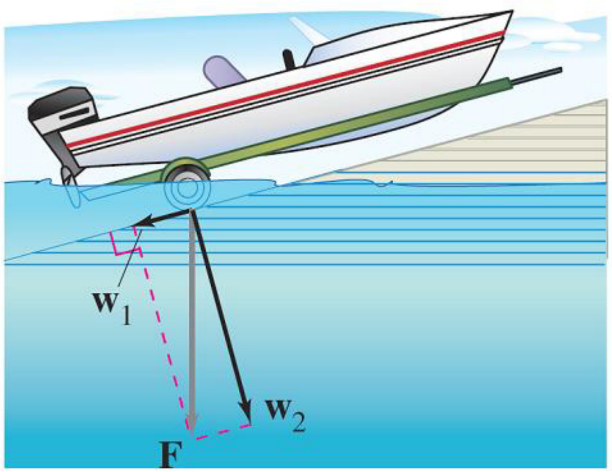


Figure 3.29

Finding Vector Components

These two orthogonal forces, $\mathbf{w}_1$ and $\mathbf{w}_2$, are vector components of $\mathbf{F}$. That is,

$$\mathbf{F} = \mathbf{w}_1 + \mathbf{w}_2.$$

Vector components of $\mathbf{F}$

The negative of component $\mathbf{w}_1$ represents the force needed to keep the boat from rolling down the ramp, whereas $\mathbf{w}_2$ represents the force that the tires must withstand against the ramp.

Finding Vector Components

A procedure for finding $\mathbf{w}_1$ and $\mathbf{w}_2$ is as shown below.

Definition of Vector Components

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors such that

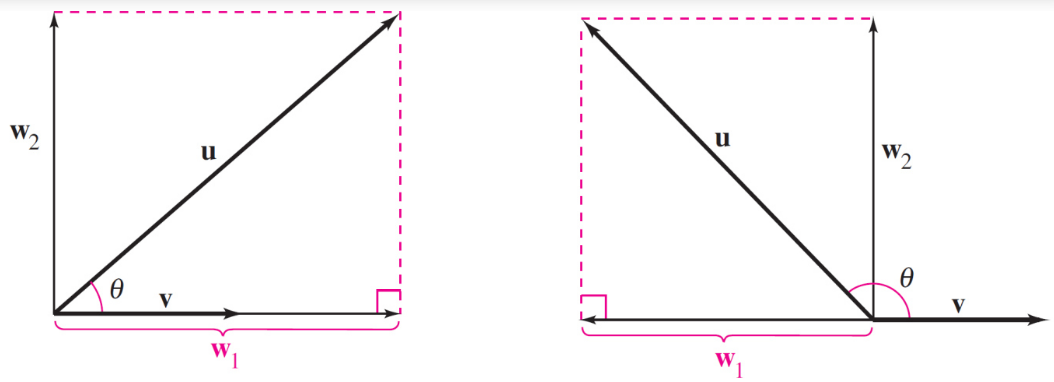
$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal and $\mathbf{w}_1$ is parallel to (or a scalar multiple of) $\mathbf{v}$, as shown in Figure 3.30. The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are called **vector components** of $\mathbf{u}$. The vector $\mathbf{w}_1$ is the **projection** of $\mathbf{u}$ onto $\mathbf{v}$ and is denoted by

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}.$$

The vector $\mathbf{w}_2$ is given by

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1.$$



θ is acute.

θ is obtuse.

Figure 3.30

Finding Vector Components

From the definition of vector components, you can see that it is easy to find the component $\mathbf{w}_2$ once you have found the projection of **u** onto **v**. To find the projection, you can use the dot product, as follows.

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

$$\mathbf{u} = c\mathbf{v} + \mathbf{w}_2$$ $\mathbf{w}_1$ is a scalar multiple of **v**.

$$\mathbf{u} \cdot \mathbf{v} = (c\mathbf{v} + \mathbf{w}_2) \cdot \mathbf{v}$$ Take dot product of each side with **v**.

$$\mathbf{u} \cdot \mathbf{v} = c\mathbf{v} \cdot \mathbf{v} + \mathbf{w}_2 \cdot \mathbf{v}$$

Finding Vector Components

$$\mathbf{u} \cdot \mathbf{v} = c\|\mathbf{v}\|^2 + 0$$

$\mathbf{w}_2$ and $\mathbf{v}$ are orthogonal.

So,

$$c = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}$$

and

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = c\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}.$$

Projection of u onto v

Let **u** and **v** be nonzero vectors. The projection of **u** onto **v** is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

Example 5 – *Decomposing a Vector into Components*

Find the projection of $\mathbf{u} = \langle 3, -5 \rangle$ onto $\mathbf{v} = \langle 6, 2 \rangle$. Then write $\mathbf{u}$ as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution:

The projection of **u** onto **v** is

$$\begin{aligned} \mathbf{w}_1 &= \text{proj}_{\mathbf{v}}\mathbf{u} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} \\ &= \left(\frac{8}{40}\right)\langle 6, 2 \rangle \end{aligned}$$

Example 5 – *Solution*

cont'd

$$= \left\langle \frac{6}{5}, \frac{2}{5} \right\rangle$$

as shown in Figure 3.31.

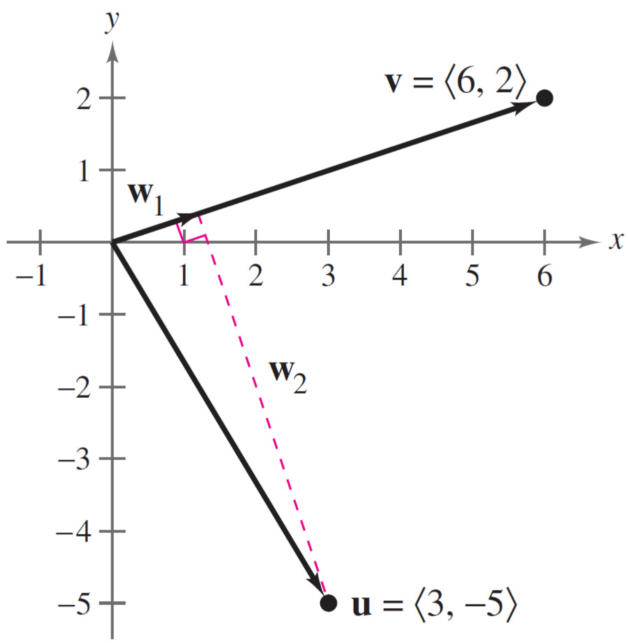


Figure 3.31

Example 5 – *Solution* cont'd

The other component $\mathbf{w_2}$, is

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$$

$$= \langle 3, -5 \rangle - \left\langle \frac{6}{5}, \frac{2}{5} \right\rangle$$

$$= \left\langle \frac{9}{5}, -\frac{27}{5} \right\rangle.$$

So,

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

$$= \left\langle \frac{6}{5}, \frac{2}{5} \right\rangle + \left\langle \frac{9}{5}, -\frac{27}{5} \right\rangle$$

$$= \langle 3, -5 \rangle.$$

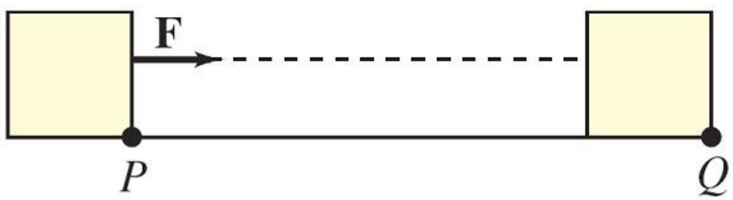
Work

Work

The work W done by a *constant* force **F** acting along the line of motion of an object is given by

$$W = \text{(magnitude of force)(distance)} = \|\mathbf{F}\| \, \|\overrightarrow{PQ}\|$$

as shown in Figure 3.33.

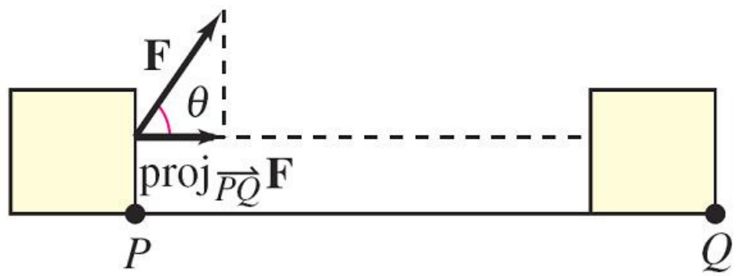


Force acts along the line of motion.

Figure 3.33

Work

When the constant force **F** is not directed along the line of motion, as shown in Figure 3.34,



Force acts at angle θ with the line of motion.

Figure 3.34

the work *W* done by the force is given by

$$W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\| \, \|\overrightarrow{PQ}\|$$

Projection form for work

$$= (\cos\theta)\|\mathbf{F}\| \, \|\overrightarrow{PQ}\|$$

$$\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\| = (\cos\theta)\|\mathbf{F}\|$$

Work

$$= \mathbf{F} \cdot \overrightarrow{PQ}.$$

Dot product form for work

This notion of work is summarized in the following definition.

Definition of Work

The **work** W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is given by either of the following.

1. $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\| \, \|\overrightarrow{PQ}\|$ Projection form

2. $W = \mathbf{F} \cdot \overrightarrow{PQ}$ Dot product form

Example 7 – *Finding Work*

To close a sliding barn door, a person pulls on a rope with a constant force of 50 pounds at a constant angle of 60°, as shown in Figure 3.35. Find the work done in moving the barn door 12 feet to its closed position.

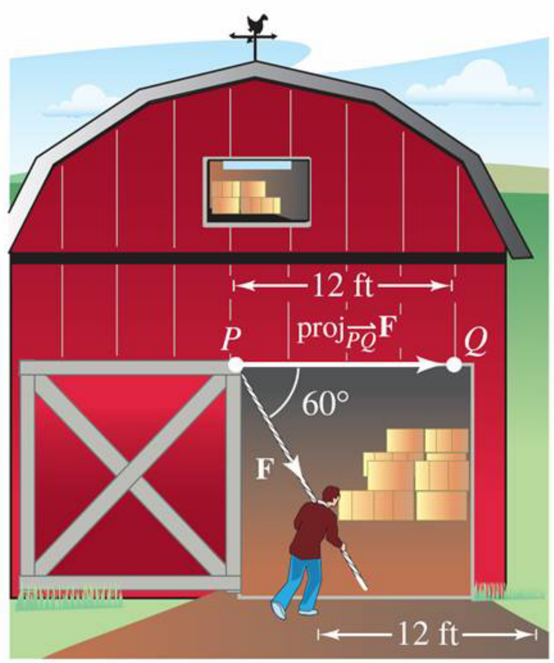


Figure 3.35

Example 7 – *Solution*

Using a projection, you can calculate the work as follows.

$$
\begin{aligned}
W &= \left\|\text{proj}_{\overrightarrow{PQ}}\,\mathbf{F}\right\| \, \left\|\overrightarrow{PQ}\right\| \\
&= (\cos 60^\circ)\|\mathbf{F}\| \, \left\|\overrightarrow{PQ}\right\| \\
&= \frac{1}{2}(50)(12) \\
&= 300 \text{ foot-pounds}
\end{aligned}
$$

So, the work done is 300 foot-pounds.

You can verify this result by finding the vectors **F** and $\overrightarrow{PQ}$ and calculating their dot product.