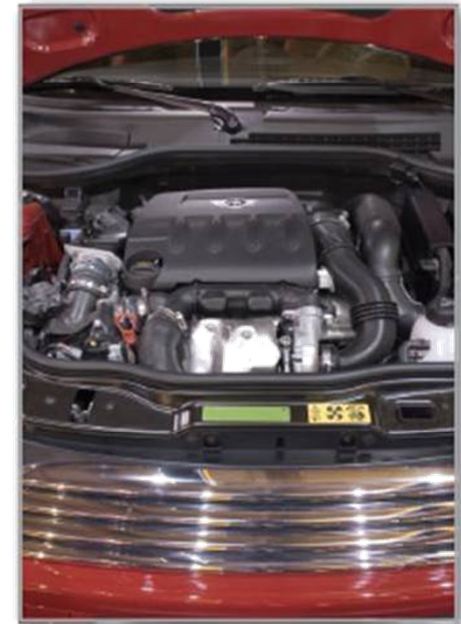


# 3 Additional Topics in Trigonometry



## 3.3

# Vectors in the Plane

# Objectives

- Represent vectors as directed line segments.
- Write the component forms of vectors.
- Perform basic vector operations and represent them graphically.
- Write vectors as linear combinations of unit vectors.
- Find the direction angles of vectors.
- Use vectors to model and solve real-life problems.



# Introduction

# Introduction

Quantities such as force and velocity involve both *magnitude* and *direction* and cannot be completely characterized by a single real number.

To represent such a quantity, you can use a **directed line segment**, as shown in Figure 3.11.

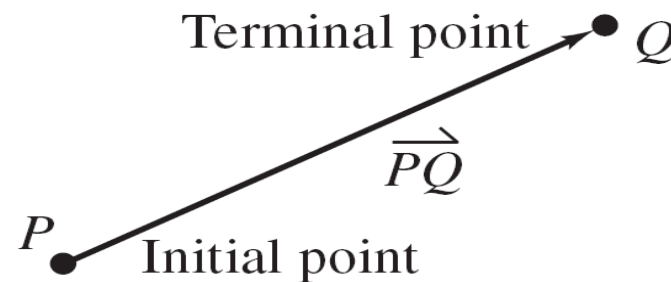


Figure 3.11

# Introduction

The directed line segment  $\overrightarrow{PQ}$  has **initial point**  $P$  and **terminal point**  $Q$ . Its **magnitude** (or length) is denoted by  $\|\overrightarrow{PQ}\|$  and can be found using the Distance Formula.

Two directed line segments that have the same magnitude and direction are equivalent.

For example, the directed line segments in Figure 3.12 are all equivalent.

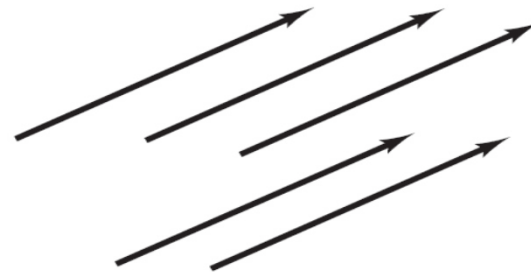


Figure 3.12

# Introduction

The set of all directed line segments that are equivalent to the directed line segment  $\overrightarrow{PQ}$  is a **vector  $\mathbf{v}$  in the plane**, written

$$\mathbf{v} = \overrightarrow{PQ}.$$

Vectors are denoted by lowercase, boldface letters such as **u**, **v**, and **w**.

## Example 1 – *Showing That Two Vectors Are Equivalent*

Show that  $\mathbf{u}$  and  $\mathbf{v}$  in Figure 3.13 are equivalent.

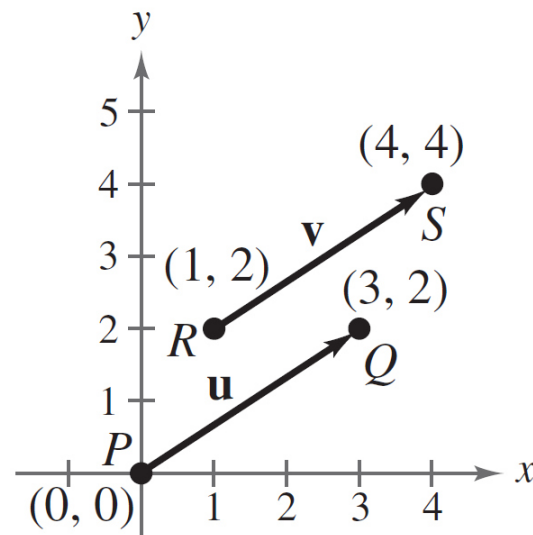


Figure 3.13



## Example 1 – *Solution*

From the Distance Formula, it follows that  $\overrightarrow{PQ}$  and  $\overrightarrow{RS}$  have the *same magnitude*.

$$\|\overrightarrow{PQ}\| = \sqrt{(3 - 0)^2 + (2 - 0)^2} = \sqrt{13}$$

$$\|\overrightarrow{RS}\| = \sqrt{(4 - 1)^2 + (4 - 2)^2} = \sqrt{13}$$

Moreover, both line segments have the *same direction* because they are both directed toward the upper right on lines having a slope of

$$\frac{4 - 2}{4 - 1} = \frac{2 - 0}{3 - 0} = \frac{2}{3}.$$

Because  $\overrightarrow{PQ}$  and  $\overrightarrow{RS}$  have the same magnitude and direction,  $\mathbf{u}$  and  $\mathbf{v}$  are equivalent.



# Component Form of a Vector

# Component Form of a Vector

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments.

This representative of the vector  $\mathbf{v}$  is in **standard position**.

A vector whose initial point is the origin  $(0, 0)$  can be uniquely represented by the coordinates of its terminal point  $(v_1, v_2)$ .

This is the **component form of a vector  $\mathbf{v}$** , written as  $\mathbf{v} = \langle v_1, v_2 \rangle$ .

# Component Form of a Vector

The coordinates  $v_1$  and  $v_2$  are the *components* of  $\mathbf{v}$ .

If both the initial point and the terminal point lie at the origin, then  $\mathbf{v}$  is the **zero vector** and is denoted by  $\mathbf{0} = \langle 0, 0 \rangle$ .

## Component Form of a Vector

The component form of the vector with initial point  $P(p_1, p_2)$  and terminal point  $Q(q_1, q_2)$  is given by

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}.$$

The **magnitude** (or length) of  $\mathbf{v}$  is given by

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

If  $\|\mathbf{v}\| = 1$ , then  $\mathbf{v}$  is a **unit vector**. Moreover,  $\|\mathbf{v}\| = 0$  if and only if  $\mathbf{v}$  is the zero vector  $\mathbf{0}$ .

# Component Form of a Vector

Two vectors  $\mathbf{u} = \langle u_1, u_2 \rangle$  and  $\mathbf{v} = \langle v_1, v_2 \rangle$  are *equal* if and only if  $u_1 = v_1$  and  $u_2 = v_2$ .

For instance, in Example 1, the vector  $\mathbf{u}$  from  $P(0, 0)$  to  $Q(3, 2)$  is  $\mathbf{u} = \overrightarrow{PQ} = \langle 3 - 0, 2 - 0 \rangle = \langle 3, 2 \rangle$ , and the vector  $\mathbf{v}$  from  $R(1, 2)$  to  $S(4, 4)$  is  $\mathbf{v} = \overrightarrow{RS} = \langle 4 - 1, 4 - 2 \rangle = \langle 3, 2 \rangle$ .

## Example 2 – Finding the Component Form of a Vector

Find the component form and magnitude of the vector  $\mathbf{v}$  that has initial point  $(4, -7)$  and terminal point  $(-1, 5)$ .

**Solution:**

Let

$$P(4, -7) = (p_1, p_2)$$

and

$$Q(-1, 5) = (q_1, q_2).$$

Then, the components of  $\mathbf{v} = \langle v_1, v_2 \rangle$  are

$$v_1 = q_1 - p_1 = -1 - 4 = -5$$

$$v_2 = q_2 - p_2 = 5 - (-7) = 12.$$

## Example 2 – *Solution*

cont'd

So,  $\mathbf{v} = \langle -5, 12 \rangle$  and the magnitude of  $\mathbf{v}$  is

$$\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2}$$

$$= \sqrt{169}$$

$$= 13.$$



# Vector Operations



# Vector Operations

The two basic vector operations are **scalar multiplication** and **vector addition**. In operations with vectors, numbers are usually referred to as **scalars**.

In this section, scalars will always be real numbers. Geometrically, the product of a vector  $\mathbf{v}$  and a scalar  $k$  is the vector that is  $|k|$  times as long as  $\mathbf{v}$ .

# Vector Operations

When  $k$  is positive,  $k\mathbf{v}$  has the same direction as  $\mathbf{v}$ , and when  $k$  is negative,  $k\mathbf{v}$  has the direction opposite that of  $\mathbf{v}$ , as shown in Figure 3.14.

To add two vectors  $\mathbf{u}$  and  $\mathbf{v}$  geometrically, first position them (without changing their lengths or directions) so that the initial point of the second vector  $\mathbf{v}$  coincides with the terminal point of the first vector  $\mathbf{u}$ .

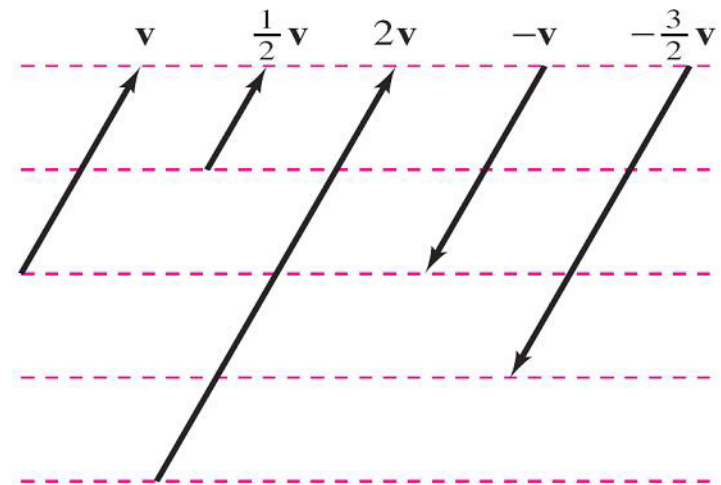
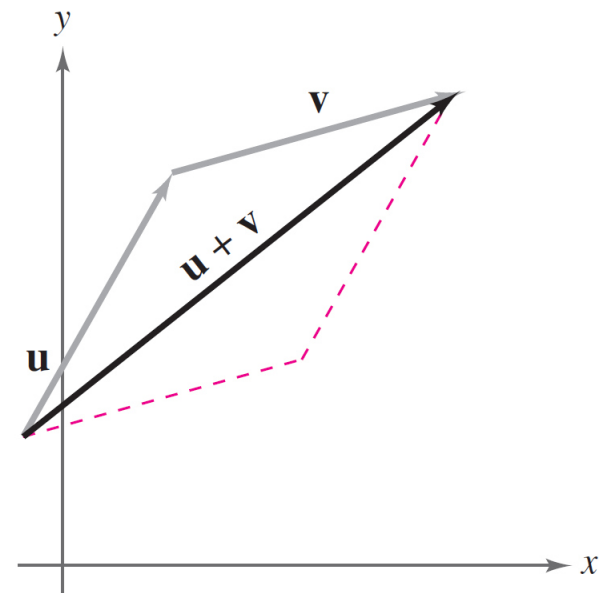
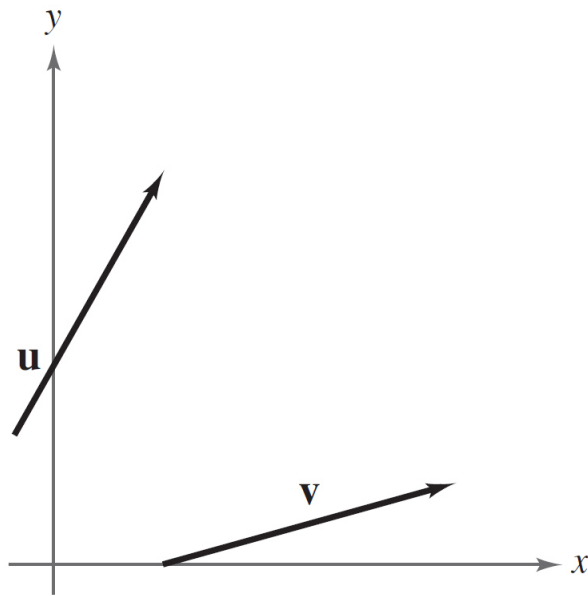


Figure 3.14

# Vector Operations

The sum  $\mathbf{u} + \mathbf{v}$  is the vector formed by joining the initial point of the first vector  $\mathbf{u}$  with the terminal point of the second vector  $\mathbf{v}$ , as shown below.



# Vector Operations

This technique is called the **parallelogram law** for vector addition because the vector  $\mathbf{u} + \mathbf{v}$ , often called the **resultant** of vector addition, is the diagonal of a parallelogram having adjacent sides  $\mathbf{u}$  and  $\mathbf{v}$ .

## Definitions of Vector Addition and Scalar Multiplication

Let  $\mathbf{u} = \langle u_1, u_2 \rangle$  and  $\mathbf{v} = \langle v_1, v_2 \rangle$  be vectors and let  $k$  be a scalar (a real number). Then the **sum** of  $\mathbf{u}$  and  $\mathbf{v}$  is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \quad \text{Sum}$$

and the **scalar multiple** of  $k$  times  $\mathbf{u}$  is the vector

$$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle. \quad \text{Scalar multiple}$$

# Vector Operations

The **negative** of  $\mathbf{v} = \langle v_1, v_2 \rangle$  is

$$-\mathbf{v} = (-1)\mathbf{v}$$

$$= \langle -v_1, -v_2 \rangle$$

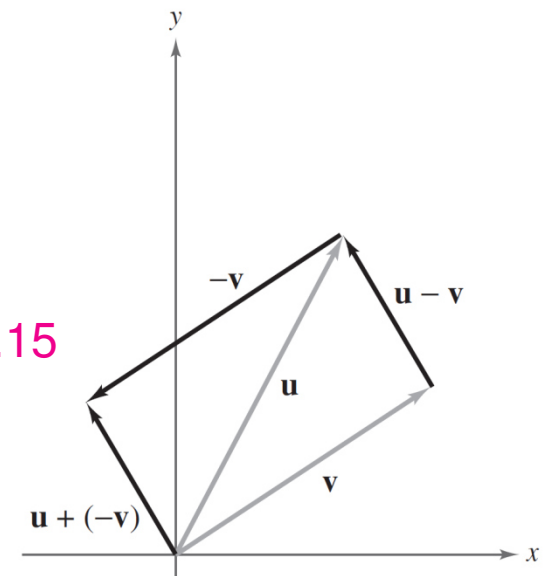
Negative

and the **difference** of  $\mathbf{u}$  and  $\mathbf{v}$  is

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

Add  $(-\mathbf{v})$ . See Figure 3.15

$$= \langle u_1 - v_1, u_2 - v_2 \rangle. \text{ Difference}$$



$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

Figure 3.15

# Vector Operations

To represent  $\mathbf{u} - \mathbf{v}$  geometrically, you can use directed line segments with the *same* initial point.

The difference  $\mathbf{u} - \mathbf{v}$  is the vector from the terminal point of  $\mathbf{v}$  to the terminal point of  $\mathbf{u}$ , which is equal to  $\mathbf{u} + (-\mathbf{v})$ , as shown in Figure 3.15.

The component definitions of vector addition and scalar multiplication are illustrated in Example 3.

In this example, notice that each of the vector operations can be interpreted geometrically.

## Example 3 – *Vector Operations*

Let  $\mathbf{v} = \langle -2, 5 \rangle$  and  $\mathbf{w} = \langle 3, 4 \rangle$ . Find each of the following vectors.

a.  $2\mathbf{v}$                       b.  $\mathbf{w} - \mathbf{v}$                       c.  $\mathbf{v} + 2\mathbf{w}$

**Solution:**

a. Because  $\mathbf{v} = \langle -2, 5 \rangle$ , you have

$$\begin{aligned} 2\mathbf{v} &= 2\langle -2, 5 \rangle \\ &= \langle 2(-2), 2(5) \rangle \\ &= \langle -4, 10 \rangle. \end{aligned}$$

A sketch of  $2\mathbf{v}$  is shown in Figure 3.16.

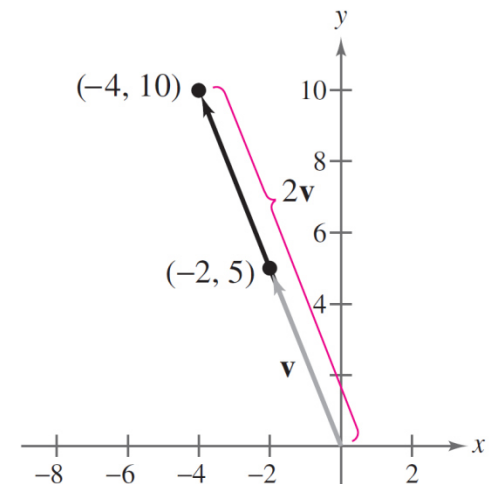


Figure 3.16

## Example 3 – *Solution*

cont'd

**b.** The difference of  $\mathbf{w}$  and  $\mathbf{v}$  is

$$\begin{aligned}\mathbf{w} - \mathbf{v} &= \langle 3, 4 \rangle - \langle -2, 5 \rangle \\ &= \langle 3 - (-2), 4 - 5 \rangle \\ &= \langle 5, -1 \rangle\end{aligned}$$

A sketch of  $\mathbf{w} - \mathbf{v}$  is shown in Figure 3.17.

Note that the figure shows the vector difference  $\mathbf{w} - \mathbf{v}$  as the sum  $\mathbf{w} + (-\mathbf{v})$ .

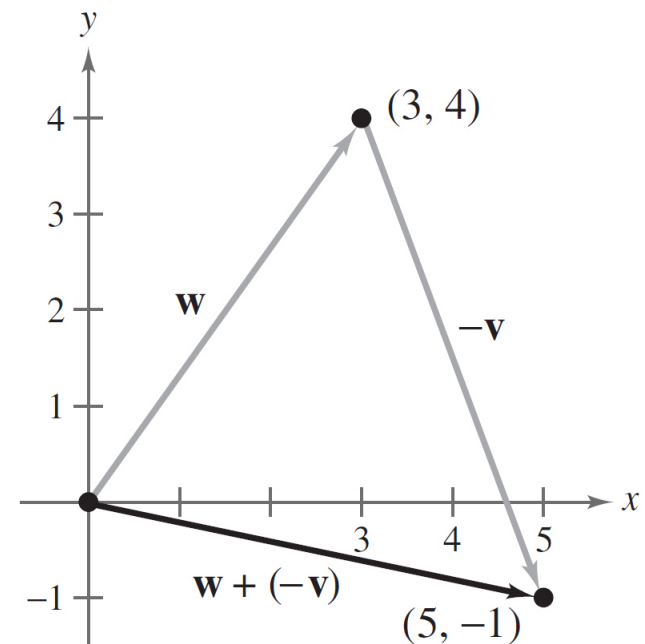


Figure 3.17



# Example 3 – Solution

cont'd

c. The sum of  $\mathbf{v}$  and  $2\mathbf{w}$  is

$$\begin{aligned}\mathbf{v} + 2\mathbf{w} &= \langle -2, 5 \rangle + 2\langle 3, 4 \rangle \\ &= \langle -2, 5 \rangle + \langle 2(3), 2(4) \rangle \\ &= \langle -2, 5 \rangle + \langle 6, 8 \rangle \\ &= \langle -2 + 6, 5 + 8 \rangle \\ &= \langle 4, 13 \rangle.\end{aligned}$$

A sketch of  $\mathbf{v} + 2\mathbf{w}$  is shown in Figure 3.18.

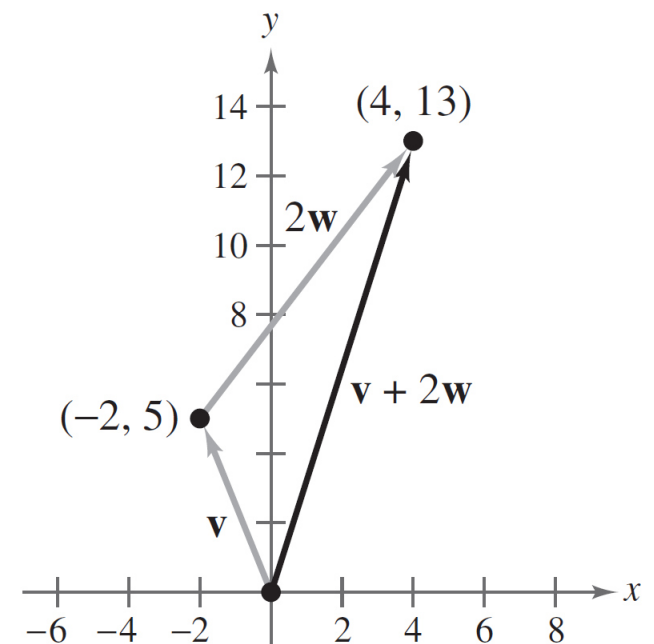


Figure 3.18

# Vector Operations

Vector addition and scalar multiplication share many of the properties of ordinary arithmetic.

## Properties of Vector Addition and Scalar Multiplication

Let  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  be vectors and let  $c$  and  $d$  be scalars. Then the following properties are true.

1.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

2.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

3.  $\mathbf{u} + \mathbf{0} = \mathbf{u}$

4.  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

5.  $c(d\mathbf{u}) = (cd)\mathbf{u}$

6.  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

7.  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

8.  $1(\mathbf{u}) = \mathbf{u}, \quad 0(\mathbf{u}) = \mathbf{0}$

9.  $\|c\mathbf{v}\| = |c| \|\mathbf{v}\|$



# Unit Vectors

# Unit Vectors

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector  $\mathbf{v}$ .

To do this, you can divide  $\mathbf{v}$  by its magnitude to obtain

$$\mathbf{u} = \text{unit vector} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}. \quad \text{Unit vector in direction of } \mathbf{v}$$

Note that  $\mathbf{u}$  is a scalar multiple of  $\mathbf{v}$ .

The vector  $\mathbf{u}$  has a magnitude of 1 and the same direction as  $\mathbf{v}$ . The vector  $\mathbf{u}$  is called a **unit vector in the direction of  $\mathbf{v}$** .

## Example 4 – *Finding a Unit Vector*

Find a unit vector in the direction of  $\mathbf{v} = \langle -2, 5 \rangle$  and verify that the result has a magnitude of 1.

**Solution:**

The unit vector in the direction of  $\mathbf{v}$  is

$$\begin{aligned}\frac{\mathbf{v}}{\|\mathbf{v}\|} &= \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + (5)^2}} \\ &= \frac{1}{\sqrt{29}} \langle -2, 5 \rangle \\ &= \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle.\end{aligned}$$

## Example 4 – *Solution*

cont'd

This vector has a magnitude of 1 because

$$\sqrt{\left(\frac{-2}{\sqrt{29}}\right)^2 + \left(\frac{5}{\sqrt{29}}\right)^2} = \sqrt{\frac{4}{29} + \frac{25}{29}}$$

$$= \sqrt{\frac{29}{29}}$$

$$= 1.$$

# Unit Vectors

The unit vectors  $\langle 1, 0 \rangle$  and  $\langle 0, 1 \rangle$  are called the **standard unit vectors** and are denoted by

$$\mathbf{i} = \langle 1, 0 \rangle \text{ and } \mathbf{j} = \langle 0, 1 \rangle$$

as shown in Figure 3.19.

(Note that the lowercase letter **i** is written in boldface to distinguish it from the imaginary unit

$$i = \sqrt{-1}.)$$

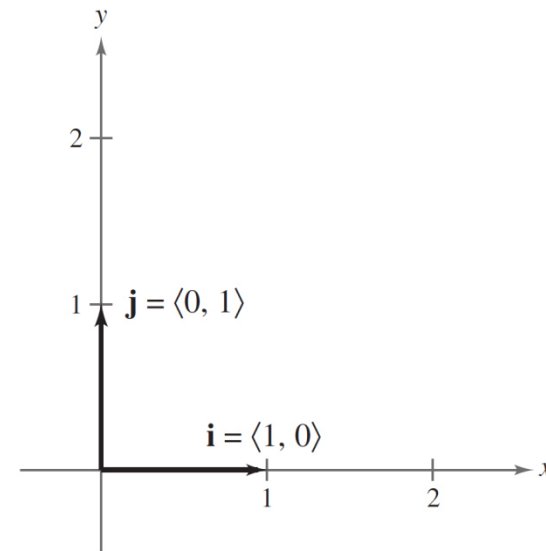


Figure 3.19

# Unit Vectors

These vectors can be used to represent any vector  $\mathbf{v} = \langle v_1, v_2 \rangle$ , as follows.

$$\begin{aligned}\mathbf{v} &= \langle v_1, v_2 \rangle \\ &= v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle \\ &= v_1 \mathbf{i} + v_2 \mathbf{j}\end{aligned}$$

The scalars  $v_1$  and  $v_2$  are called the **horizontal** and **vertical components of  $\mathbf{v}$** , respectively.



# Unit Vectors

The vector sum

$$v_1\mathbf{i} + v_2\mathbf{j}$$

is called a **linear combination** of the vectors  $\mathbf{i}$  and  $\mathbf{j}$ .

Any vector in the plane can be written as a linear combination of the standard unit vectors  $\mathbf{i}$  and  $\mathbf{j}$ .



# Direction Angles

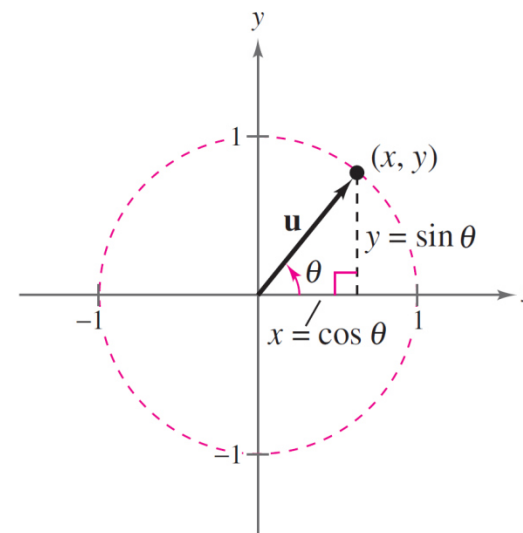
# Direction Angles

If  $\mathbf{u}$  is a *unit vector* such that  $\theta$  is the angle (measured counterclockwise) from the positive  $x$ -axis to  $\mathbf{u}$ , then the terminal point of  $\mathbf{u}$  lies on the unit circle and you have

$$\begin{aligned}\mathbf{u} &= \langle x, y \rangle = \langle \cos \theta, \sin \theta \rangle \\ &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}\end{aligned}$$

as shown in Figure 3.21.

The angle  $\theta$  is the **direction angle** of the vector  $\mathbf{u}$ .



$$\|\mathbf{u}\| = 1$$

Figure 3.21

# Direction Angles

Suppose that  $\mathbf{u}$  is a unit vector with direction angle  $\theta$ .

If  $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$  is any vector that makes an angle  $\theta$  with the positive  $x$ -axis, then it has the same direction as  $\mathbf{u}$  and you can write

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\|\langle \cos \theta, \sin \theta \rangle \\ &= \|\mathbf{v}\|(\cos \theta)\mathbf{i} + \|\mathbf{v}\|(\sin \theta)\mathbf{j}.\end{aligned}$$

# Direction Angles

Because  $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \|\mathbf{v}\|(\cos \theta)\mathbf{i} + \|\mathbf{v}\|(\sin \theta)\mathbf{j}$ , it follows that the direction angle  $\theta$  for  $\mathbf{v}$  is determined from

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

Quotient identity

$$= \frac{\|\mathbf{v}\| \sin \theta}{\|\mathbf{v}\| \cos \theta}$$

Multiply numerator and denominator by  $\|\mathbf{v}\|$ .

$$= \frac{b}{a}$$

Simplify.

## Example 8 – Finding Direction Angles of Vectors

Find the direction angle of each vector.

**a.**  $\mathbf{u} = 3\mathbf{i} + 3\mathbf{j}$

**b.**  $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

**Solution:**

**a.** The direction angle is determined from

$$\tan \theta = \frac{b}{a} = \frac{3}{3} = 1.$$

So,  $\theta = 45^\circ$ , as shown in Figure 3.22.

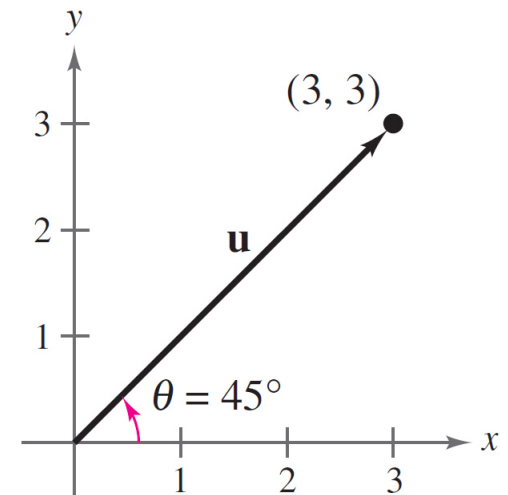


Figure 3.22

## Example 8 – *Solution*

cont'd

**b.** The direction angle is determined from

$$\tan \theta = \frac{b}{a} = \frac{-4}{3}.$$

Moreover, because  $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$  lies in Quadrant IV,  $\theta$  lies in Quadrant IV, and its reference angle is

$$\begin{aligned}\theta' &= \left| \arctan\left(-\frac{4}{3}\right) \right| \approx \left| -0.9273 \text{ radian} \right| \\ &\approx \left| -53.13^\circ \right| \\ &= 53.13^\circ.\end{aligned}$$

# Example 8 – *Solution*

cont'd

So, it follows that  $\theta \approx 360^\circ - 53.13^\circ = 306.87^\circ$ , as shown in Figure 3.23.

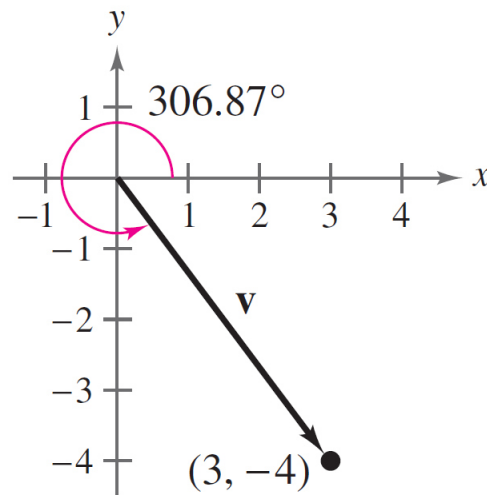


Figure 3.23





# Applications

## Example 9 – Finding the Component Form of a Vector

Find the component form of the vector that represents the velocity of an airplane descending at a speed of 150 miles per hour at an angle  $20^\circ$  below the horizontal, as shown in Figure 3.24.

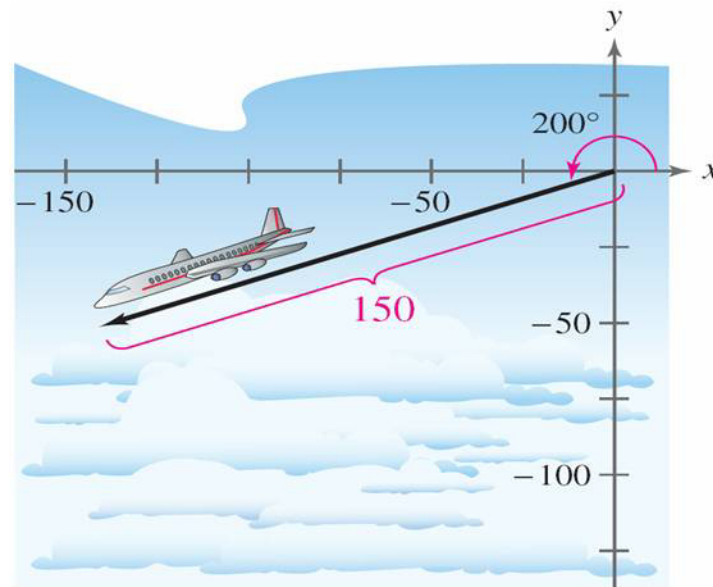


Figure 3.24

## Example 9 – *Solution*

The velocity vector  $\mathbf{v}$  has a magnitude of 150 and a direction angle of  $\theta = 200^\circ$ .

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\|(\cos \theta)\mathbf{i} + \|\mathbf{v}\|(\sin \theta)\mathbf{j} \\ &= 150(\cos 200^\circ)\mathbf{i} + 150(\sin 200^\circ)\mathbf{j} \\ &\approx 150(-0.9397)\mathbf{i} + 150(-0.3420)\mathbf{j} \\ &\approx -140.96\mathbf{i} - 51.30\mathbf{j} \\ &= \langle -140.96, -51.30 \rangle\end{aligned}$$

## Example 9 – *Solution*

cont'd

You can check that  $\mathbf{v}$  has a magnitude of 150, as follows.

$$\|\mathbf{v}\| \approx \sqrt{(-140.96)^2 + (-51.30)^2}$$

$$= \sqrt{22,501.41}$$

$$\approx 150$$