3 Additional Topics in Trigonometry

3.2 Law of Cosines

Objectives

- Use the Law of Cosines to solve oblique triangles (SSS or SAS).

- Use the Law of Cosines to model and solve real-life problems.

- Use Heron's Area Formula to find the area of a triangle.

Introduction

Introduction

Two cases remain in the list of conditions needed to solve an oblique triangle—SSS and SAS.

When you are given three sides (SSS), or two sides and their included angle (SAS), none of the ratios in the Law of Sines would be complete.

In such cases, you can use the **Law of Cosines.**

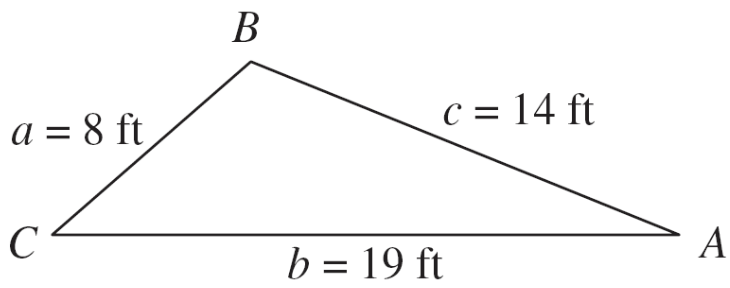
Introduction

Law of Cosines

Standard Form	Alternative Form
$a^2 = b^2 + c^2 - 2bc \cos A$	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
$b^2 = a^2 + c^2 - 2ac \cos B$	$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
$c^2 = a^2 + b^2 - 2ab \cos C$	$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Example 1 – *Three Sides of a Triangle—SSS*

Find the three angles of the triangle shown below.



Solution:

It is a good idea first to find the angle opposite the longest side—side *b* in this case. Using the alternative form of the Law of Cosines, you find that

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

Example 1 – *Solution* cont'd

$$= \frac{8^2 + 14^2 - 19^2}{2(8)(14)}$$

$$\approx -0.45089.$$

Because cos *B* is negative, *B* is an *obtuse* angle given by *B* ≈ 116.80°.

At this point, it is simpler to use the Law of Sines to determine *A*.

$$\sin A = a\left(\frac{\sin B}{b}\right)$$

Example 1 – *Solution*

cont'd

$$\approx 8\left(\frac{\sin 116.80°}{19}\right)$$

$$\approx 0.37583$$

Because B is obtuse and a triangle can have at most one obtuse angle, you know that A must be acute.

So, $A \approx 22.08°$ and

$$C \approx 180° - 22.08° - 116.80°$$

$$= 41.12°.$$

Introduction

Do you see why it was wise to find the largest angle *first* in Example 1? Knowing the cosine of an angle, you can determine whether the angle is acute or obtuse. That is,

$\cos\theta > 0$	for	$0° < \theta < 90°$	Acute
$\cos\theta < 0$	for	$90° < \theta < 180°$.	Obtuse

So, in Example 1, once you found that angle *B* was obtuse, you knew that angles *A* and *C* were both acute.

Furthermore, if the largest angle is acute, then the remaining two angles are also acute.

Applications

Example 3 – *An Application of the Law of Cosines*

The pitcher's mound on a women's softball field is 43 feet from home plate and the distance between the bases is 60 feet, as shown in Figure 3.9. (The pitcher's mound is *not* halfway between home plate and second base.) How far is the pitcher's mound from first base?

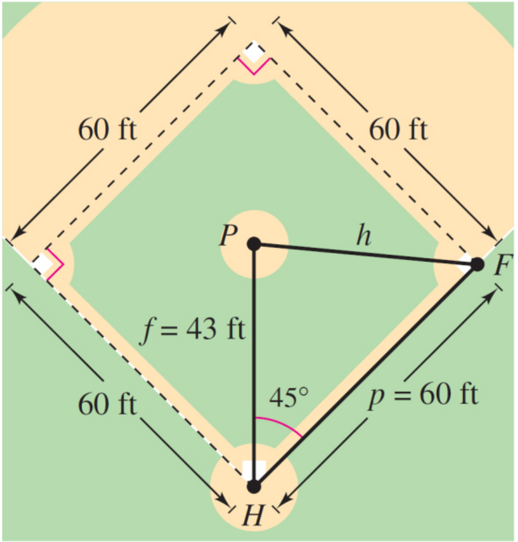


Figure 3.9

Example 3 – *Solution*

In triangle *HPF*, $H = 45°$ (line *HP* bisects the right angle at *H*), $f = 43$, and $p = 60$.

Using the Law of Cosines for this SAS case, you have

$$h^2 = f^2 + p^2 - 2fp \cos H$$

$$= 43^2 + 60^2 - 2(43)(60) \cos 45°$$

$$\approx 1800.3.$$

So, the approximate distance from the pitcher's mound to first base is

$$h \approx \sqrt{1800.3} \approx 42.43 \text{ feet.}$$

Heron’s Area Formula

Heron’s Area Formula

The Law of Cosines can be used to establish the following formula for the area of a triangle.

This formula is called **Heron’s Area Formula** after the Greek mathematician Heron.

Heron’s Area Formula

Given any triangle with sides of lengths a, b, and c, the area of the triangle is

$$\text{Area} = \sqrt{s(s - a)(s - b)(s - c)}$$

where

$$s = \frac{a + b + c}{2}.$$

Example 5 – *Using Heron's Area Formula*

Find the area of a triangle having sides of lengths $a = 43$ meters, $b = 53$ meters, and $c = 72$ meters.

Solution:

Because $s = (a + b + c)/2$

$= 168/2$

$= 84,$

Heron's Area Formula yields

$$\text{Area} = \sqrt{s(s - a)(s - b)(s - c)}$$

Example 5 – *Solution*

cont'd

$$= \sqrt{84(84 - 43)(84 - 53)(84 - 72)}$$

$$= \sqrt{84(41)(31)(12)}$$

$\approx$ 1131.89 square meters.

Heron's Area Formula

You have now studied three different formulas for the area of a triangle.

Standard Formula: Area $= \frac{1}{2}bh$

Oblique Triangle:

$$\text{Area} = \tfrac{1}{2}bc \sin A$$
$$= \tfrac{1}{2}ab \sin C$$
$$= \tfrac{1}{2}ac \sin B$$

Heron's Area Formula: Area $= \sqrt{s(s-a)(s-b)(s-c)}$