2 Analytic Trigonometry

2.5 Multiple-Angle and Product-to-Sum Formulas

Objectives

- Use multiple-angle formulas to rewrite and evaluate trigonometric functions.
- Use power-reducing formulas to rewrite and evaluate trigonometric functions.
- Use half-angle formulas to rewrite and evaluate trigonometric functions.

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- Use product-to-sum and sum-to-product formulas to rewrite and evaluate trigonometric functions.

- Use trigonometric formulas to rewrite real-life models.

Multiple-Angle Formulas

Multiple-Angle Formulas

In this section, you will study four other categories of trigonometric identities.

1. The first category involves *functions of multiple angles* such as sin ku and cos ku.

2. The second category involves *squares of trigonometric functions* such as $\sin^2 u$.

3. The third category involves *functions of half-angles* such as $\sin(u/2)$.

4. The fourth category involves *products of trigonometric functions* such as sin u cos v.

Multiple-Angle Formulas

You should learn the **double-angle formulas** because they are used often in trigonometry and calculus.

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

$$\cos 2u = \cos^2 u - \sin^2 u$$

$$= 2 \cos^2 u - 1$$

$$= 1 - 2 \sin^2 u$$

Example 1 – *Solving a Multiple-Angle Equation*

Solve $2 \cos x + \sin 2x = 0$.

Solution:

Begin by rewriting the equation so that it involves functions of x (rather than $2x$). Then factor and solve.

$2 \cos x + \sin 2x = 0$	Write original equation.
$2 \cos x + 2 \sin x \cos x = 0$	Double-angle formula.
$2 \cos x(1 + \sin x) = 0$	Factor.
$2 \cos x = 0$ and $1 + \sin x = 0$	Set factors equal to zero.

Example 1 – *Solution*

cont'd

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad\qquad x = \frac{3\pi}{2}$$

Solutions in [0, 2π)

So, the general solution is

$$x = \frac{\pi}{2} + 2n\pi \quad \text{and} \quad x = \frac{3\pi}{2} + 2n\pi$$

where *n* is an integer. Try verifying these solutions graphically.

Power-Reducing Formulas

Power-Reducing Formulas

The double-angle formulas can be used to obtain the following **power-reducing formulas.**

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Example 4 – *Reducing a Power*

Rewrite $\sin^4 x$ in terms of first powers of the cosines of multiple angles.

Solution:

Note the repeated use of power-reducing formulas.

$$\sin^4 x = (\sin^2 x)^2 \qquad \text{Property of exponents}$$

$$= \left(\frac{1 - \cos 2x}{2}\right)^2 \qquad \text{Power-reducing formula}$$

$$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \qquad \text{Expand.}$$

Example 4 – *Solution*

cont'd

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right)$$

Power-reducing formula

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x$$

Distributive Property

$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

Simplify.

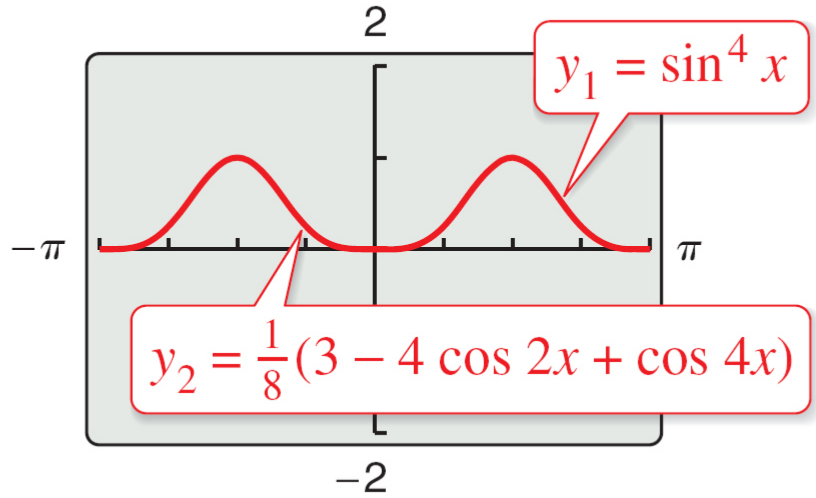
$$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$

Factor out common factor.

Example 4 – *Solution*

cont'd

You can use a graphing utility to check this result, as shown below. Notice that the graphs coincide.



Half-Angle Formulas

Half-Angle Formulas

You can derive some useful alternative forms of the power-reducing formulas by replacing u with $u/2$. The results are called **half-angle formulas.**

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The signs of $\sin\frac{u}{2}$ and $\cos\frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ lies.

Example 5 – *Using a Half-Angle Formula*

Find the exact value of sin 105°.

Solution:

Begin by noting that 105° is half of 210°. Then, using the half-angle formula for sin(u/2) and the fact that 105° lies in Quadrant II, you have

$$\sin 105^\circ = \sqrt{\frac{1 - \cos 210^\circ}{2}}$$

$$= \sqrt{\frac{1 + \left(\sqrt{3}/2\right)}{2}}$$

Example 5 – *Solution*

cont'd

$$= \frac{\sqrt{2 + \sqrt{3}}}{2}.$$

The positive square root is chosen because sin θ is positive in Quadrant II.

Product-to-Sum Formulas

Product-to-Sum Formulas

Each of the following **product-to-sum formulas** can be verified using the sum and difference formulas.

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

Product-to-sum formulas are used in calculus to solve problems involving the products of sines and cosines of two different angles.

Example 7 – *Writing Products as Sums*

Rewrite the product $\cos 5x \sin 4x$ as a sum or difference.

Solution:

Using the appropriate product-to-sum formula, you obtain

$$\cos 5x \sin 4x = \tfrac{1}{2}[\sin(5x + 4x) - \sin(5x - 4x)]$$

$$= \tfrac{1}{2}\sin 9x - \tfrac{1}{2}\sin x.$$

Product-to-Sum Formulas

Occasionally, it is useful to reverse the procedure and write a sum of trigonometric functions as a product. This can be accomplished with the following **sum-to-product formulas.**

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Application

Example 10 – *Projectile Motion*

Ignoring air resistance, the range of a projectile fired at an angle θ with the horizontal and with an initial velocity of v_0 feet per second is given by

$$r = \frac{1}{16}v_0^2 \sin\theta \cos\theta$$

where r is the horizontal distance (in feet) that the projectile travels.

Example 10 – *Projectile Motion* cont'd

A football player can kick a football from ground level with an initial velocity of 80 feet per second

a. Write the projectile motion model in a simpler form.

b. At what angle must the player kick the football so that the football travels 200 feet?

Example 10 – *Solution*

a. You can use a double-angle formula to rewrite the projectile motion model as

$$r = \frac{1}{32} v_0^2 (2 \sin \theta \cos \theta)$$ Rewrite original projectile motion model.

$$= \frac{1}{32} v_0^2 \sin 2\theta.$$ Rewrite model using a double-angle formula.

b. $$r = \frac{1}{32} v_0^2 \sin 2\theta$$ Write projectile motion model.

$$200 = \frac{1}{32} (80)^2 \sin 2\theta$$ Substitute 200 for r and 80 for v_0.

Example 10 – *Solution*

cont'd

$200 = 200 \sin 2\theta$ Simplify.

$1 = \sin 2\theta$ Divide each side by 200.

You know that $2\theta = \pi/2$, so dividing this result by 2 produces $\theta = \pi/4$.

Because $\pi/4 = 45°$, the player must kick the football at an angle of 45° so that the football travels 200 feet.