3 Additional Topics in Trigonometry

3.1 Law of Sines

Objectives

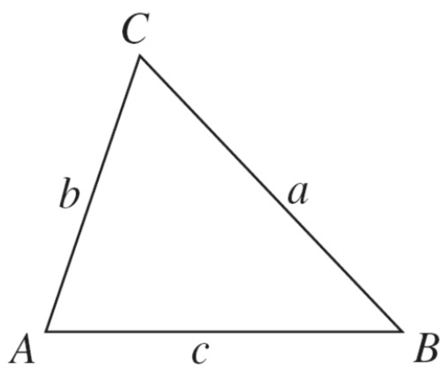
- Use the Law of Sines to solve oblique triangles (AAS or ASA).
- Use the Law of Sines to solve oblique triangles (SSA).
- Find the areas of oblique triangles.
- Use the Law of Sines to model and solve real-life problems.

Introduction

Introduction

In this section, you will solve **oblique triangles**—triangles that have no right angles.

As standard notation, the angles of a triangle are labeled *A*, *B*, and *C*, and their opposite sides are labeled *a*, *b*, and *c*, as shown on the right.



To solve an oblique triangle, you need to know the measure of at least one side and any two other measures of the triangle—either two sides, two angles, or one angle and one side.

Introduction

This breaks down into the following four cases.

1. Two angles and any side (AAS or ASA)

2. Two sides and an angle opposite one of them (SSA)

3. Three sides (SSS)

4. Two sides and their included angle (SAS)

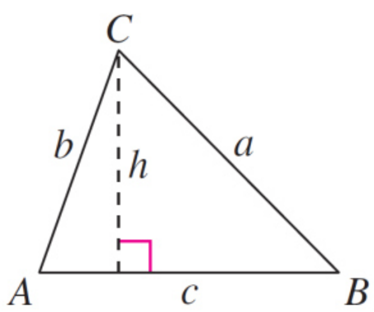
The first two cases can be solved using the **Law of Sines,** whereas the last two cases require the Law of Cosines.

Introduction

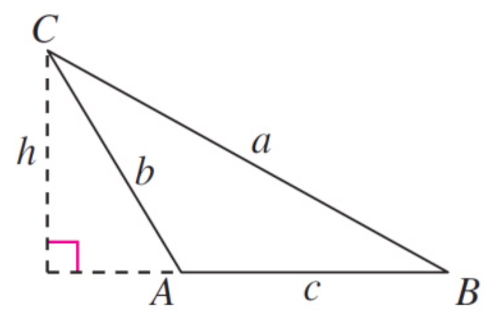
Law of Sines

If ABC is a triangle with sides a, b, and c, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



A is acute.



A is obtuse.

The Law of Sines can also be written in the reciprocal form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Example 1 – *Given Two Angles and One Side—AAS*

For the triangle in Figure 3.1, $C = 102°$, $B = 29°$, and $b = 28$ feet. Find the remaining angle and sides.

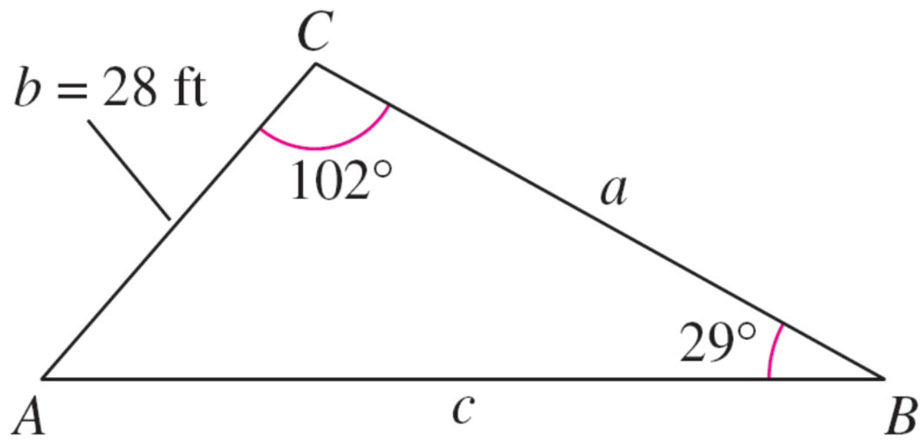


Figure 3.1

Example 1 – *Solution*

The third angle of the triangle is

$$A = 180° - B - C$$

$$= 180° - 29° - 102°$$

$$= 49°.$$

By the Law of Sines, you have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Example 1 – *Solution*

cont'd

Using b = 28 produces

$$a = \frac{b}{\sin B}(\sin A) = \frac{28}{\sin 29^\circ}(\sin 49^\circ) \approx 43.59 \text{ feet}$$

and

$$c = \frac{b}{\sin B}(\sin C) = \frac{28}{\sin 29^\circ}(\sin 102^\circ) \approx 56.49 \text{ feet.}$$

The Ambiguous Case (SSA)

The Ambiguous Case (SSA)

In Example 1, you saw that two angles and one side determine a unique triangle.

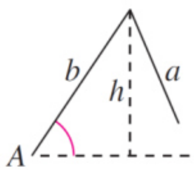
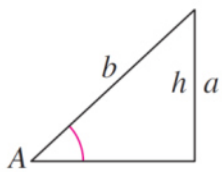
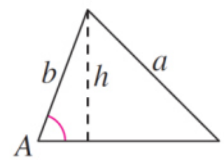
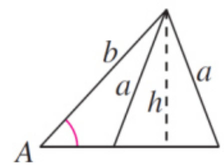
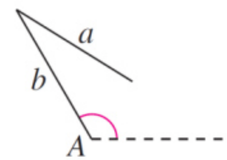
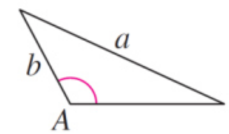
However, if two sides and one opposite angle are given, three possible situations can occur:

(1) no such triangle exists,
(2) one such triangle exists, or
(3) two distinct triangles may satisfy the conditions.

The Ambiguous Case (SSA)

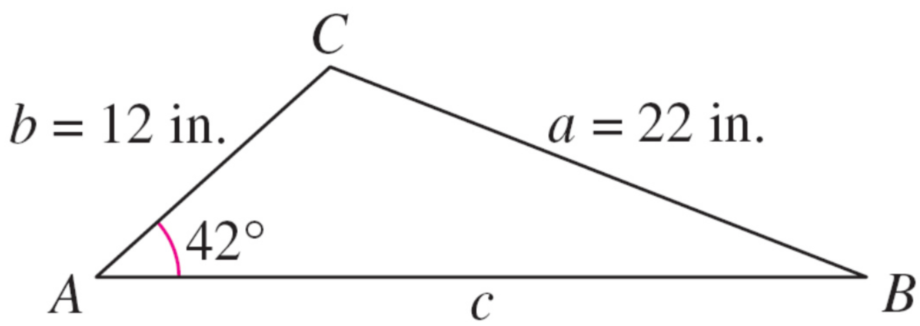
The Ambiguous Case (SSA)

Consider a triangle in which you are given a, b, and A. $(h = b \sin A)$

	A is acute.	A is acute.	A is acute.	A is acute.	A is obtuse.	A is obtuse.
Sketch						
Necessary condition	$a < h$	$a = h$	$a \geq b$	$h < a < b$	$a \leq b$	$a > b$
Triangles possible	None	One	One	Two	None	One

Example 3 – *Single-Solution Case—SSA*

For the triangle in Figure 3.4, a = 22 inches, b = 12 inches, and A = 42°. Find the remaining side and angles.



One solution: $a \geq b$

Figure 3.4

Example 3 – *Solution*

By the Law of Sines, you have

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$ Reciprocal form

$$\sin B = b\left(\frac{\sin A}{a}\right)$$ Multiply each side by *b*.

$$\sin B = 12\left(\frac{\sin 42^\circ}{22}\right)$$ Substitute for *A*, *a*, and *b*.

$$B \approx 21.41^\circ.$$

Example 3 – *Solution*

cont'd

Now, you can determine that

$$C \approx 180° - 42° - 21.41°$$

$$= 116.59°.$$

Then, the remaining side is

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

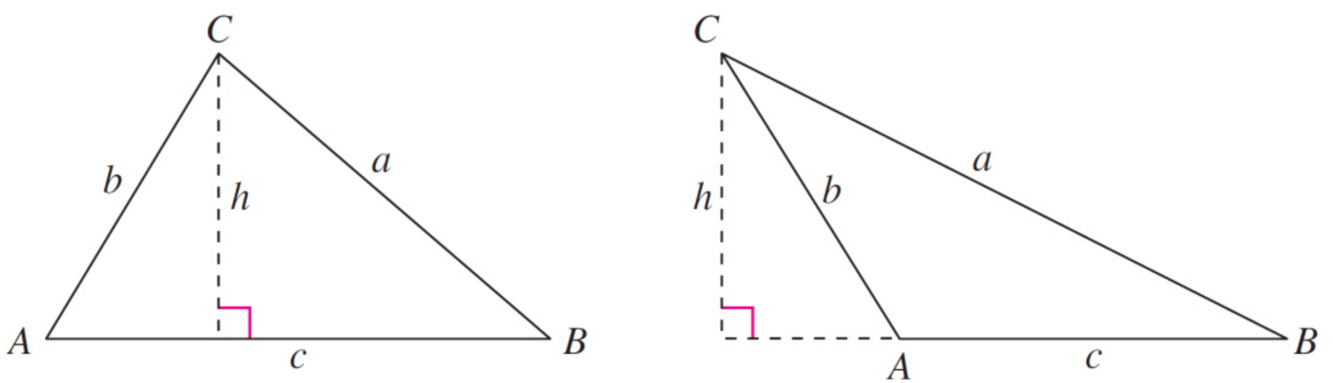
$$c = \frac{a}{\sin A}(\sin C) = \frac{22}{\sin 42°}(\sin 116.59°) \approx 29.40 \text{ inches.}$$

Area of an Oblique Triangle

Area of an Oblique Triangle

The procedure used to prove the Law of Sines leads to a simple formula for the area of an oblique triangle.

Referring to triangles below, note that each triangle has a height of $h = b \sin A$.



A is acute. *A* is obtuse.

Area of an Oblique Triangle

Consequently, the area of each triangle is

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(c)(b \sin A) = \frac{1}{2} bc \sin A.$$

By similar arguments, you can develop the formulas

$$\text{Area} = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B.$$

Area of an Oblique Triangle

Area of an Oblique Triangle

The area of any triangle is one-half the product of the lengths of two sides times the sine of their included angle. That is,

$$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B.$$

Note that when angle A is 90°, the formula gives the area of a right triangle:

$$\text{Area} = \frac{1}{2}bc(\sin 90°) = \frac{1}{2}bc = \frac{1}{2}(\text{base})(\text{height}).$$

$\sin 90° = 1$

Similar results are obtained for angles C and B equal to 90°.

Example 6 – *Finding the Area of a Triangular Lot*

Find the area of a triangular lot having two sides of lengths 90 meters and 52 meters and an included angle of 102°.

Solution:

Consider a = 90 meters, b = 52 meters, and angle C = 102°, as shown in Figure 3.6.

Then, the area of the triangle is

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$= \frac{1}{2}(90)(52)(\sin 102°)$$

$$\approx 2289 \text{ square meters.}$$

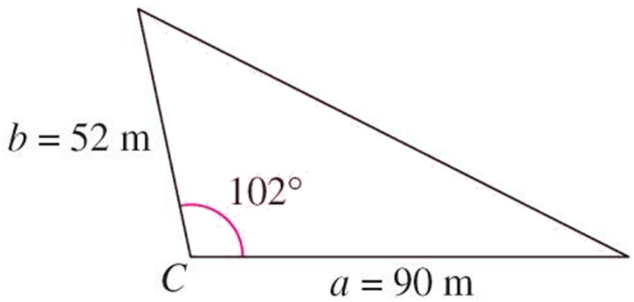


Figure 3.6

Application

Example 7 – *An Application of the Law of Sines*

The course for a boat race starts at point *A* and proceeds in the direction S 52° W to point *B*, then in the direction S 40° E to point *C*, and finally back to *A*, as shown in Figure 3.7. Point *C* lies 8 kilometers directly south of point *A*. Approximate the total distance of the race course.

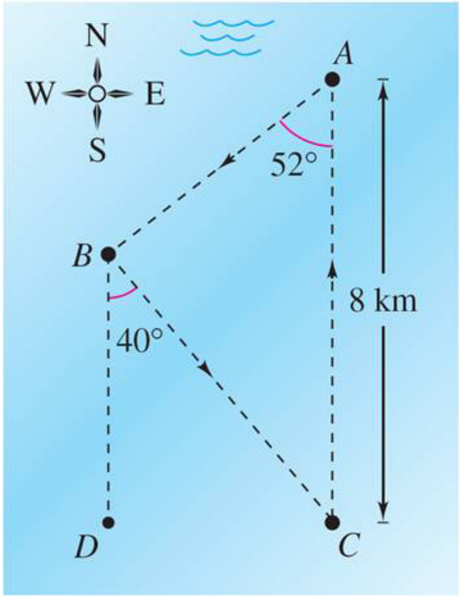


Figure 3.7

Example 7 – *Solution*

Because lines *BD* and *AC* are parallel, it follows that $\angle BCA \cong \angle CBD$.

Consequently, triangle *ABC* has the measures shown in Figure 3.8.

The measure of angle *B* is $180° – 52° – 40° = 88°$.

Using the Law of Sines,

$$\frac{a}{\sin 52°} = \frac{b}{\sin 88°} = \frac{c}{\sin 40°}.$$

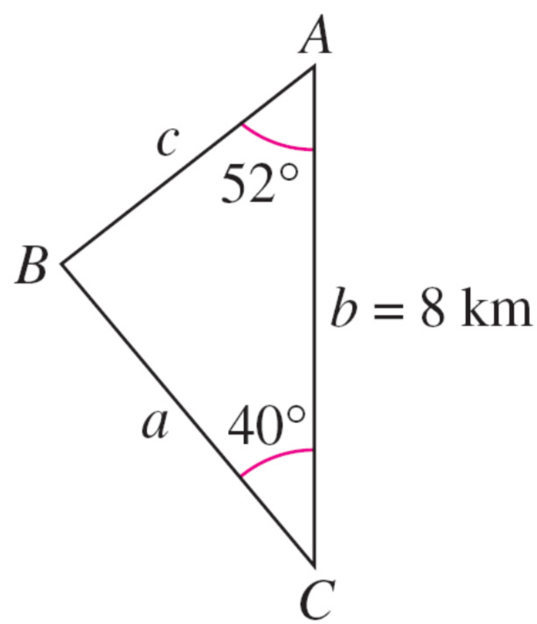


Figure 3.8

Example 7 – *Solution*

cont'd

Because $b = 8$,

$$a = \frac{8}{\sin 88^\circ}(\sin 52^\circ) \approx 6.31$$

and

$$c = \frac{8}{\sin 88^\circ}(\sin 40^\circ) \approx 5.15.$$

The total distance of the course is approximately

Length ≈ 8 + 6.31 + 5.15

=19.46 kilometers.