2 Analytic Trigonometry

2.3 Solving Trigonometric Equations

Objectives

- Use standard algebraic techniques to solve trigonometric equations.
- Solve trigonometric equations of quadratic type.
- Solve trigonometric equations involving multiple angles.
- Use inverse trigonometric functions to solve trigonometric equations.

Introduction

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To solve a trigonometric equation, use standard algebraic techniques (when possible) such as collecting like terms and factoring.

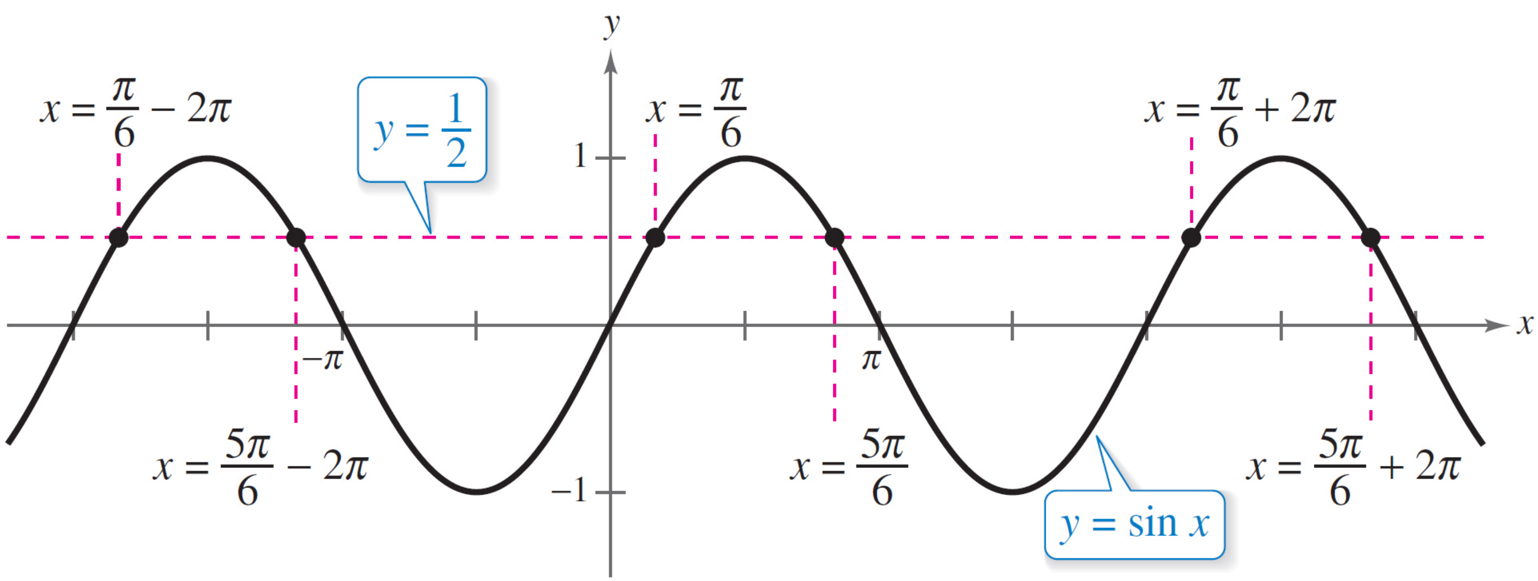
Your preliminary goal in solving a trigonometric equation is to *isolate* the trigonometric function on one side of the equation.

For example, to solve the equation 2 sin x = 1, divide each side by 2 to obtain

$$\sin x = \frac{1}{2}.$$

Introduction

To solve for *x*, note in the figure below that the equation $\sin x = \frac{1}{2}$ has solutions $x = \pi/6$ and $x = 5\pi/6$ in the interval $[0, 2\pi)$.



Introduction

Moreover, because sin *x* has a period of 2π, there are infinitely many other solutions, which can be written as

$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi$$

General solution

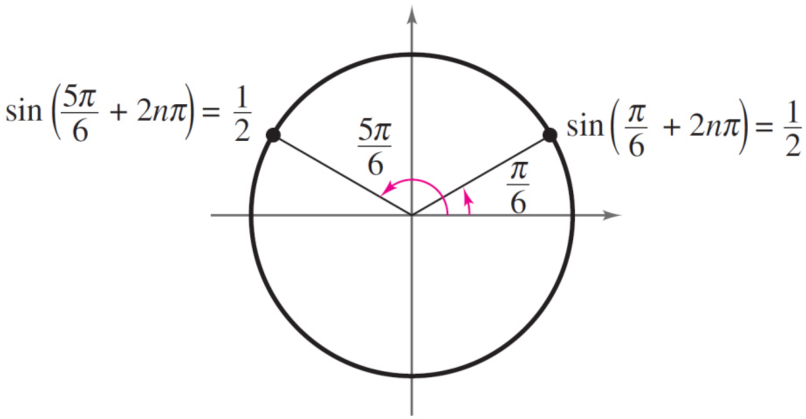
where *n* is an integer, as shown above.

Introduction

The figure below illustrates another way to show that the equation $\sin x = \frac{1}{2}$ has infinitely many solutions.

Any angles that are coterminal with $\pi/6$ or $5\pi/6$ will also be solutions of the equation.

When solving trigonometric equations, you should write your answer(s) using exact values rather than decimal approximations.



Example 1 – *Collecting Like Terms*

Solve $\sin x + \sqrt{2} = -\sin x.$

Solution:

Begin by isolating sin *x* on one side of the equation.

$$\sin x + \sqrt{2} = -\sin x$$ Write original equation.

$$\sin x + \sin x + \sqrt{2} = 0$$ Add sin *x* to each side.

$$\sin x + \sin x = -\sqrt{2}$$ Subtract $\sqrt{2}$ from each side.

Example 1 – *Solution*

cont'd

$$2 \sin x = -\sqrt{2}$$ Combine like terms.

$$\sin x = -\frac{\sqrt{2}}{2}$$ Divide each side by 2.

Because sin *x* has a period of 2π, first find all solutions in the interval $[0, 2\pi)$.

These solutions are $x = 5\pi/4$ and $x = 7\pi/4$. Finally, add multiples of 2π to each of these solutions to obtain the general form

$$x = \frac{5\pi}{4} + 2n\pi \quad \text{and} \quad x = \frac{7\pi}{4} + 2n\pi$$ General solution

where *n* is an integer.

Equations of Quadratic Type

Equations of Quadratic Type

Many trigonometric equations are of quadratic type $ax^2 + bx + c = 0$, as shown below.

Quadratic in sin *x*	**Quadratic in sec *x***
$2\sin^2 x - \sin x - 1 = 0$	$\sec^2 x - 3\sec x - 2 = 0$
$2(\sin x)^2 - \sin x - 1 = 0$	$(\sec x)^2 - 3(\sec x) - 2 = 0$

To solve equations of this type, factor the quadratic or, when this is not possible, use the Quadratic Formula.

Example 4 – *Factoring an Equation of Quadratic Type*

Find all solutions of $2\sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi)$.

Solution:

Treat the equation as a quadratic in $\sin x$ and factor.

$2\sin^2 x - \sin x - 1 = 0$ Write original equation.

$(2\sin x + 1)(\sin x - 1) = 0$ Factor.

Example 4 – *Solution*

cont'd

Setting each factor equal to zero, you obtain the following solutions in the interval $[0, 2\pi)$.

$2 \sin x + 1 = 0$ and $\sin x - 1 = 0$

$$\sin x = -\frac{1}{2} \qquad \sin x = 1$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6} \qquad x = \frac{\pi}{2}$$

Functions Involving Multiple Angles

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The next example involve trigonometric functions of multiple angles of the forms cos *ku* and tan *ku*.

To solve equations of these forms, first solve the equation for *ku*, and then divide your result by *k*.

Example 7 – *Solving a Multiple-Angle Equation*

Solve $2 \cos 3t - 1 = 0$.

Solution:

$$2 \cos 3t - 1 = 0$$ Write original equation.

$$2 \cos 3t = 1$$ Add 1 to each side.

$$\cos 3t = \frac{1}{2}$$ Divide each side by 2.

Example 7 – *Solution*

cont'd

In the interval $[0, 2\pi)$, you know that $3t = \pi/3$ and $3t = 5\pi/3$ are the only solutions, so, in general, you have

$$3t = \frac{\pi}{3} + 2n\pi \quad \text{and} \quad 3t = \frac{5\pi}{3} + 2n\pi.$$

Dividing these results by 3, you obtain the general solution

$$t = \frac{\pi}{9} + \frac{2n\pi}{3} \quad \text{and} \quad t = \frac{5\pi}{9} + \frac{2n\pi}{3}$$

General solution

where n is an integer.

Using Inverse Functions

Example 9 – *Using Inverse Functions*

$\sec^2 x - 2\tan x = 4$	Original equation
$1 + \tan^2 x - 2\tan x - 4 = 0$	Pythagorean identity
$\tan^2 x - 2\tan x - 3 = 0$	Combine like terms.
$(\tan x - 3)(\tan x + 1) = 0$	Factor.

Setting each factor equal to zero, you obtain two solutions in the interval $(-\pi/2, \pi/2)$. [We know that the range of the inverse tangent function is $(-\pi/2, \pi/2)$.]

$x = \arctan 3$ and $x = \arctan(-1) = -\pi/4$

Example 9 – *Using Inverse Functions* cont'd

Finally, because tan x has a period of π, you add multiples of π to obtain

$$x = \arctan 3 + n\pi \quad \text{and} \quad x = (-\pi/4) + n\pi$$ General solution

where n is an integer.

You can use a calculator to approximate the value of arctan 3.