1 Trigonometry

1.7

Inverse Trigonometric Functions

Objectives

- Evaluate and graph the inverse sine function.
- Evaluate and graph the other inverse trigonometric functions.
- Evaluate the compositions of trigonometric functions.

Inverse Sine Function

Inverse Sine Function

We know that, for a function to have an inverse function, it must be one-to-one—that is, it must pass the Horizontal Line Test.

From Figure 1.51, you can see that $y = \sin x$ does not pass the test because different values of x yield the same y-value.

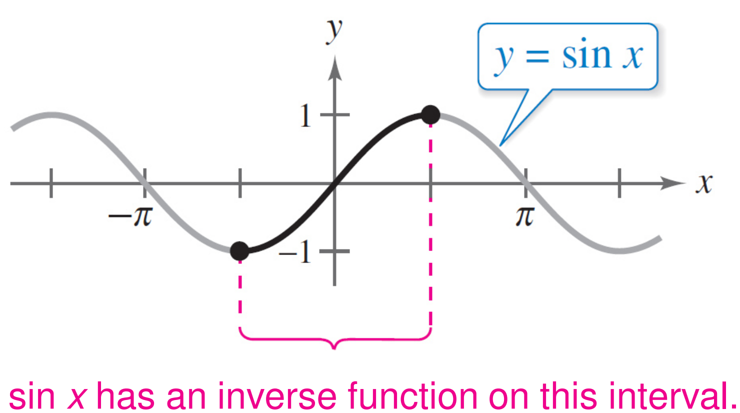


Figure 1.51

Inverse Sine Function

However, when you restrict the domain to the interval $-\pi/2 \leq x \leq \pi/2$ (corresponding to the black portion of the graph in Figure 1.51), the following properties hold.

1. On the interval $[-\pi/2, \pi/2]$, the function $y = \sin x$ is increasing.

2. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ takes on its full range of values, $-1 \leq \sin x \leq 1$.

3. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ is one-to-one.

Inverse Sine Function

So, on the restricted domain $-\pi/2 \leq x \leq \pi/2$, $y = \sin x$ has a unique inverse function called the **inverse sine function.** It is denoted by

$$y = \arcsin x \quad \text{or} \quad y = \sin^{-1} x.$$

The notation $\sin^{-1} x$ is consistent with the inverse function notation $f^{-1}(x)$.

The arcsin x notation (read as "the arcsine of x") comes from the association of a central angle with its intercepted *arc length* on a unit circle.

So, arcsin x means the angle (or arc) whose sine is x.

Inverse Sine Function

Both notations, arcsin *x* and $\sin^{-1} x$, are commonly used in mathematics, so remember that $\sin^{-1} x$ denotes the *inverse* sine function rather than 1/sin *x*.

The values of arcsin *x* lie in the interval

$$-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}.$$

The graph of *y* = arcsin *x* is shown in Figure 1.52.

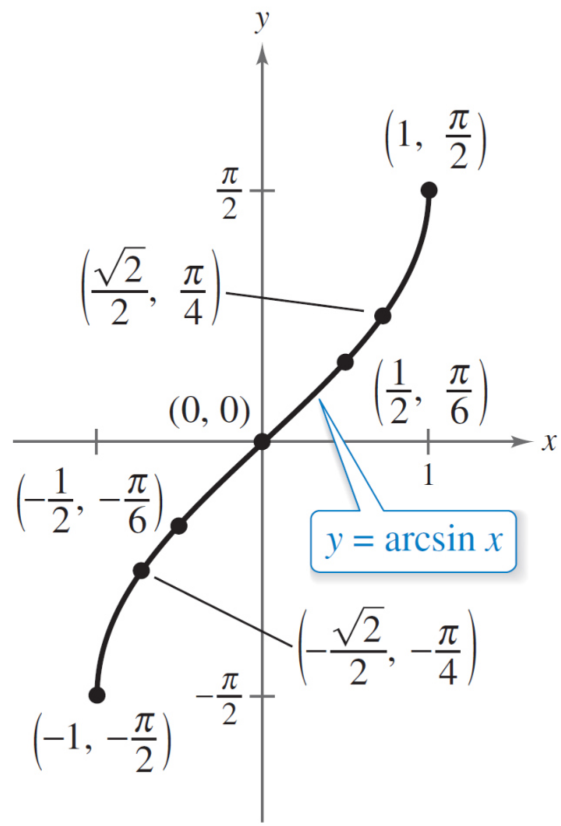


Figure 1.52

Inverse Sine Function

Definition of Inverse Sine Function

The **inverse sine function** is defined by

$$y = \arcsin x \quad \text{if and only if} \quad \sin y = x$$

where $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$. The domain of $y = \arcsin x$ is $[-1, 1]$, and the range is $[-\pi/2, \pi/2]$.

Example 1 – *Evaluating the Inverse Sine Function*

If possible, find the exact value.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\sin^{-1}\frac{\sqrt{3}}{2}$ **c.** $\sin^{-1} 2$

Solution:

a. Because $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

it follows that

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

Angle whose sine is $-\frac{1}{2}$

Example 1 – *Solution*

cont'd

b. Because $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

it follows that

$$\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}.$$

Angle whose sine is $\sqrt{3}/2$

c. It is not possible to evaluate $y = \sin^{-1} x$ when $x = 2$ because there is no angle whose sine is 2.

Remember that the domain of the inverse sine function is [–1, 1].

Example 2 – *Graphing the Arcsine Function*

Sketch a graph of $y = \arcsin x$.

Solution:

By definition, the equations $y = \arcsin x$ and $\sin y = x$ are equivalent for $-\pi/2 \le y \le \pi/2$. So, their graphs are the same.

From the interval $[-\pi/2, \pi/2]$, you can assign values to y in the equation $\sin y = x$ to make a table of values.

y	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x = \sin y$	-1	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	1

Example 2 – *Solution*

cont'd

Then plot the points and connect them with a smooth curve.

The resulting graph for $y = \arcsin x$ is shown in Figure 1.52.

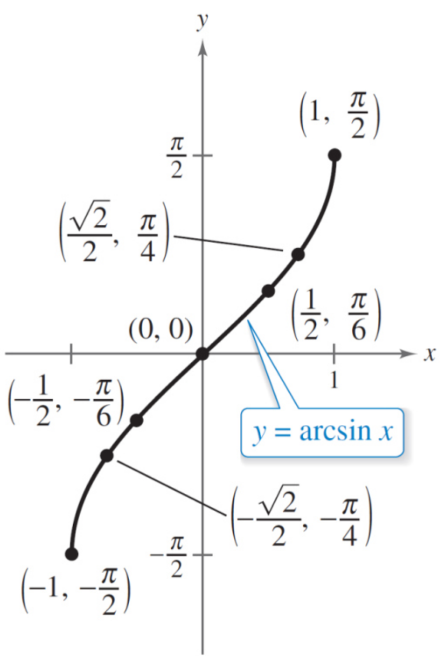
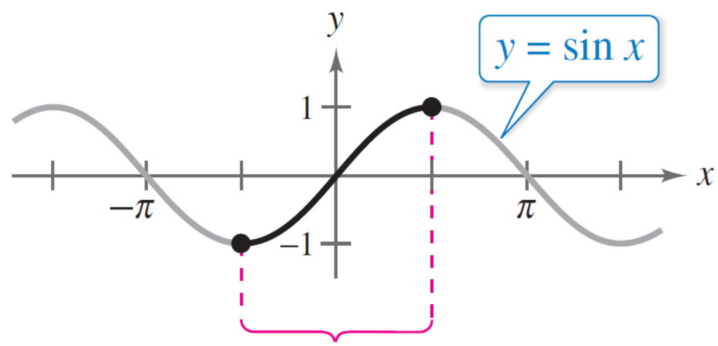


Figure 1.52

Example 2 – *Solution* cont'd

Note that it is the reflection (in the line $y = x$) of the black portion of the graph in Figure 1.51.



sin *x* has an inverse function on this interval.

Figure 1.51

Be sure you see that Figure 1.52 shows the *entire* graph of the inverse sine function.

Example 2 – *Solution*

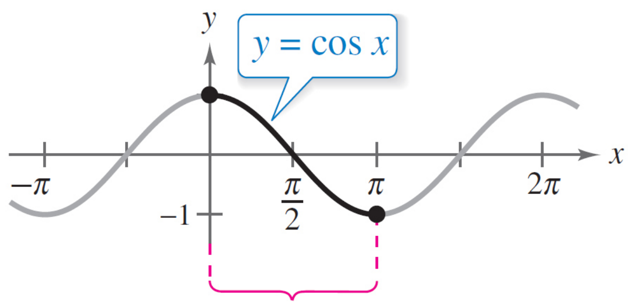
cont'd

Remember that the domain of y = arcsin x is the closed interval [–1, 1] and the range is the closed interval [$-\pi/2$, $\pi/2$].

Other Inverse Trigonometric Functions

Other Inverse Trigonometric Functions

The cosine function is decreasing and one-to-one on the interval $0 \leq x \leq \pi$, as shown below.



cos x has an inverse function on this interval.

Consequently, on this interval the cosine function has an inverse function—the **inverse cosine function**—denoted by

$$y = \arccos x \quad \text{or} \quad y = \cos^{-1} x.$$

Other Inverse Trigonometric Functions

Similarly, you can define an **inverse tangent function** by restricting the domain of $y = \tan x$ to the interval $(-\pi/2, \pi/2)$.

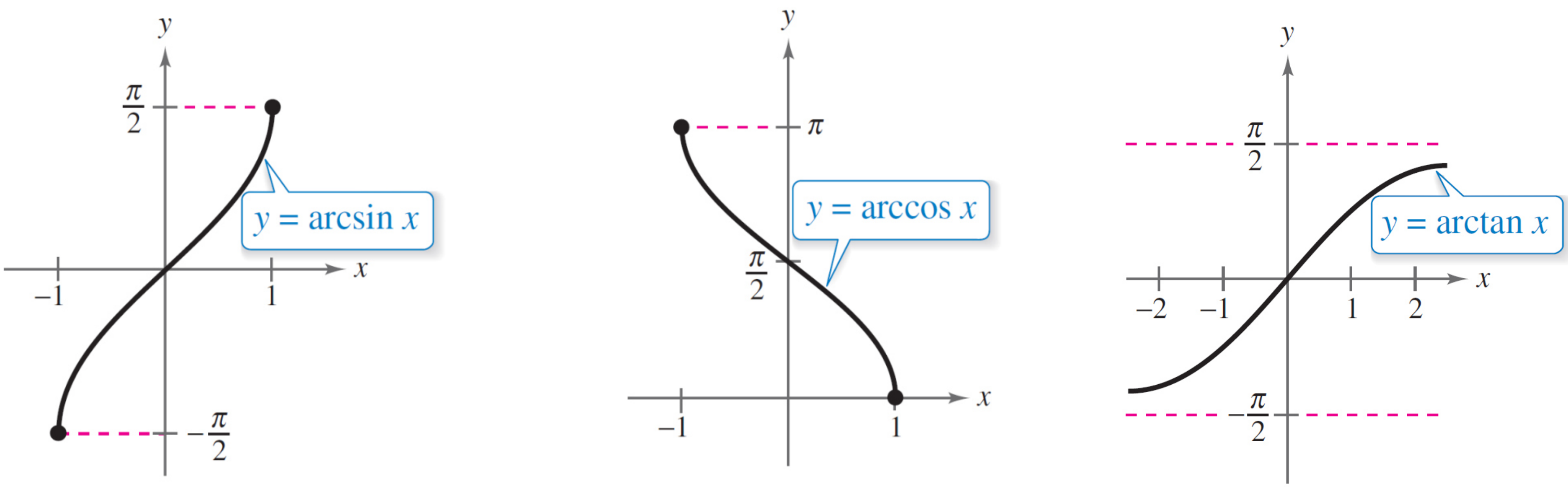
The following list summarizes the definitions of the three most common inverse trigonometric functions.

Definitions of the Inverse Trigonometric Functions

Function	Domain	Range
$y = \arcsin x$ if and only if $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ if and only if $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ if and only if $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

Other Inverse Trigonometric Functions

The graphs of three inverse trigonometric functions are shown below.



Domain: [–1,1]
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Domain: [–1,1]
Range: $[0, \pi]$

Domain: $(-\infty, \infty)$
Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example 3 – *Evaluating Inverse Trigonometric Functions*

Find the exact value.

a. arccos $\frac{\sqrt{2}}{2}$

b. arctan 0

c. $\tan^{-1}(-1)$

Solution:

a. Because cos $(\pi/4) = \sqrt{2}/2$, and $\pi/4$ lies in $[0, \pi]$, it follows that

$$\arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}.$$

Angle whose cosine is $\sqrt{2}/2$

Example 3 – *Solution*

cont'd

b. Because tan 0 = 0, and 0 lies in $(-\pi/2, \pi/2)$, it follows that

$$\arctan 0 = 0.$$

Angle whose tangent is 0

c. Because $\tan(-\pi/4) = -1$, and $-\pi/4$ lies in $(-\pi/2, \pi/2)$, it follows that

$$\tan^{-1}(-1) = -\frac{\pi}{4}.$$

Angle whose tangent is –1

Other Inverse Trigonometric Functions

By definition, the values of inverse trigonometric functions are *always in radians*.

Compositions of Functions

Compositions of Functions

We know that for all x in the domains of f and f^{-1}, inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Inverse Properties of Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-1 \le x \le 1$ and $0 \le y \le \pi$, then

$$\cos(\arccos x) = x \quad \text{and} \quad \arccos(\cos y) = y.$$

If x is a real number and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

Compositions of Functions

These inverse properties do not apply for arbitrary values of x and y. For instance,

$$\arcsin\left(\sin\frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2} \neq \frac{3\pi}{2}.$$

In other words, the property

$$\arcsin(\sin y) = y$$

is not valid for values of y outside the interval $[-\pi/2, \pi/2]$.

Example 5 – *Using Inverse Properties*

If possible, find the exact value.

a. tan[arctan(–5)] **b.** $\arcsin\left(\sin \frac{5\pi}{3}\right)$ **c.** cos($\cos^{-1}\pi$)

Solution:

a. Because –5 lies in the domain of the arctangent function, the inverse property applies, and you have

tan[arctan(–5)] = –5.

Example 5 – *Solution*

cont'd

b. In this case, $5\pi/3$ does not lie in the range of the arcsine function, $-\pi/2 \leq y \leq \pi/2$.

However, $5\pi/3$ is coterminal with

$$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$$

which does lie in the range of the arcsine function, and you have

$$\arcsin\left(\sin\frac{5\pi}{3}\right) = \arcsin\left[\sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}.$$

Example 5 – *Solution*

cont'd

c. The expression $\cos(\cos^{-1}\pi)$ is not defined because $\cos^{-1}\pi$ is not defined.

Remember that the domain of the inverse cosine function is [–1, 1].