1 Trigonometry

1.6 Graphs of Other Trigonometric Functions

Objectives

- Sketch the graphs of tangent functions.
- Sketch the graphs of cotangent functions.
- Sketch the graphs of secant and cosecant functions.
- Sketch the graphs of damped trigonometric functions.

Graph of the Tangent Function

Graph of the Tangent Function

You know that the tangent function is odd. That is, $\tan(-x) = -\tan x$. Consequently, the graph of $y = \tan x$ is symmetric with respect to the origin.

You also know from the identity $\tan x = \sin x/\cos x$ that the tangent is undefined for values at which $\cos x = 0$.

Two such values are $x = \pm\pi/2 \approx \pm 1.5708$.

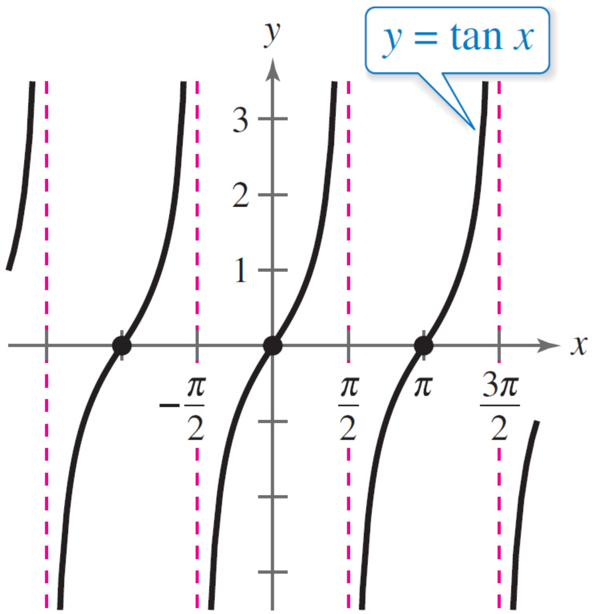
Graph of the Tangent Function

x	$-\frac{\pi}{2}$	-1.57	-1.5	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	1.5	1.57	$\frac{\pi}{2}$
$\tan x$	Undef.	-1255.8	-14.1	-1	0	1	14.1	1255.8	Undef.

As indicated in the table, tan x increases without bound as x approaches $\pi/2$ from the left and decreases without bound as x approaches $-\pi/2$ from the right.

Graph of the Tangent Function

So, the graph of $y = \tan x$ has *vertical asymptotes* at $x = \pi/2$ and $x = -\pi/2$, as shown below.



Period: π

Domain: all $x \neq \dfrac{\pi}{2} + n\pi$

Range: $(-\infty, \infty)$

Vertical asymptotes: $x = \dfrac{\pi}{2} + n\pi$

Symmetry: origin

Graph of the Tangent Function

Moreover, because the period of the tangent function is π, vertical asymptotes also occur at $x = \pi/2 + n\pi$, where n is an integer.

The domain of the tangent function is the set of all real numbers other than $x = \pi/2 + n\pi$, and the range is the set of all real numbers.

Sketching the graph of $y = a \tan(bx - c)$ is similar to sketching the graph of $y = a \sin(bx - c)$ in that you locate key points that identify the intercepts and asymptotes.

Graph of the Tangent Function

Two consecutive vertical asymptotes can be found by solving the equations

$$bx - c = -\frac{\pi}{2} \qquad \text{and} \qquad bx - c = \frac{\pi}{2}.$$

The midpoint between two consecutive vertical asymptotes is an x-intercept of the graph.

The period of the function $y = a \tan(bx - c)$ is the distance between two consecutive vertical asymptotes.

Graph of the Tangent Function

The amplitude of a tangent function is not defined.

After plotting the asymptotes and the *x*-intercept, plot a few additional points between the two asymptotes and sketch one cycle.

Finally, sketch one or two additional cycles to the left and right.

Example 1 – *Sketching the Graph of a Tangent Function*

Sketch the graph of $y = \tan \frac{x}{2}$.

Solution:

By solving the equations

$$\frac{x}{2} = -\frac{\pi}{2} \qquad \text{and} \qquad \frac{x}{2} = \frac{\pi}{2}$$

$$x = -\pi \qquad\qquad x = \pi$$

you can see that two consecutive vertical asymptotes occur at $x = -\pi$ and $x = \pi$.

Example 1 – *Solution*

cont'd

Between these two asymptotes, plot a few points, including the *x*-intercept, as shown in the table.

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
$\tan \frac{x}{2}$	Undef.	-1	0	1	Undef.

Example 1 – *Solution*

cont'd

Three cycles of the graph are shown in Figure 1.44.

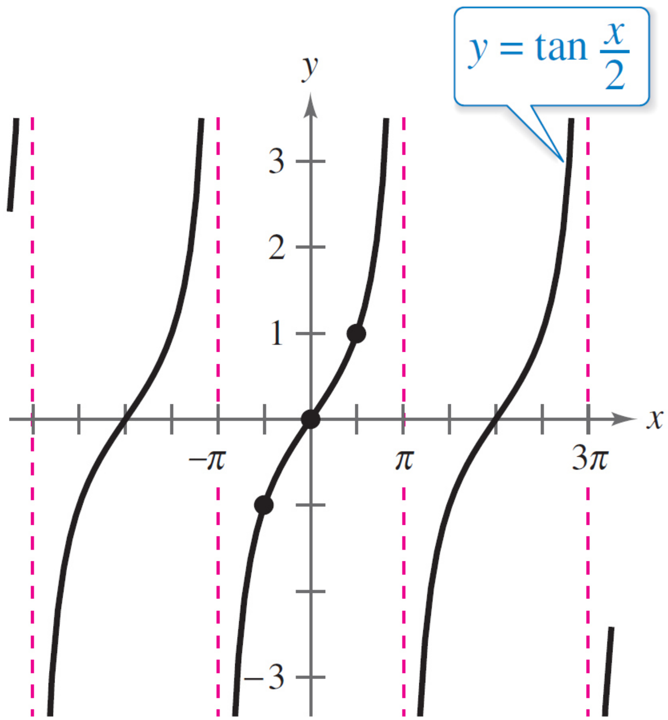


Figure 1.44

Graph of the Cotangent Function

Graph of the Cotangent Function

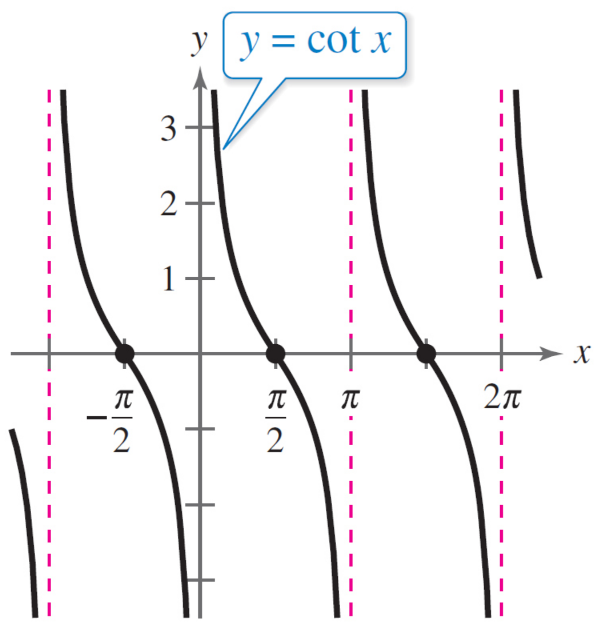
The graph of the cotangent function is similar to the graph of the tangent function. It also has a period of π. However, from the identity

$$y = \cot x = \frac{\cos x}{\sin x}$$

you can see that the cotangent function has vertical asymptotes when $\sin x$ is zero, which occurs at $x = n\pi$, where n is an integer.

Graph of the Cotangent Function

The graph of the cotangent function is shown below. Note that two consecutive vertical asymptotes of the graph of $y = a \cot(bx - c)$ can be found by solving the equations $bx - c = 0$ and $bx - c = \pi$.



Period: π
Domain: all $x \neq n\pi$
Range: $(-\infty, \infty)$
Vertical asymptotes: $x = n\pi$
Symmetry: origin

Example 3 – *Sketching the Graph of a Cotangent Function*

Sketch the graph of $y = 2 \cot \frac{x}{3}$.

Solution:

By solving the equations

$$\frac{x}{3} = 0 \qquad\qquad \frac{x}{3} = \pi$$

$$x = 0 \qquad\qquad x = 3\pi$$

you can see that two consecutive vertical asymptotes occur at $x = 0$ and $x = 3\pi$.

Example 3 – *Solution*

cont'd

Between these two asymptotes, plot a few points, including the *x*-intercept, as shown in the table.

x	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	3π
$2\cot\frac{x}{3}$	Undef.	2	0	-2	Undef.

Example 3 – *Solution*

cont'd

Three cycles of the graph are shown in Figure 1.46. Note that the period is 3π, the distance between consecutive asymptotes.

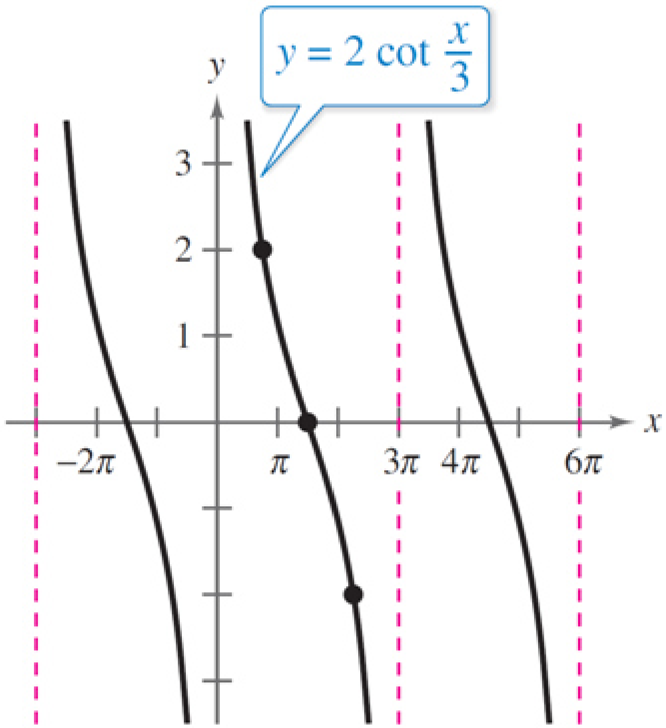


Figure 1.46

Graphs of the Reciprocal Functions

Graphs of the Reciprocal Functions

You can obtain the graphs of the two remaining trigonometric functions from the graphs of the sine and cosine functions using the reciprocal identities

$$\csc x = \frac{1}{\sin x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}.$$

For instance, at a given value of *x*, the *y*-coordinate of sec *x* is the reciprocal of the *y*-coordinate of cos *x*.

Of course, when cos $x = 0$, the reciprocal does not exist. Near such values of *x*, the behavior of the secant function is similar to that of the tangent function.

Graphs of the Reciprocal Functions

In other words, the graphs of

$$\tan x = \frac{\sin x}{\cos x} \qquad \text{and} \qquad \sec x = \frac{1}{\cos x}$$

have vertical asymptotes where cos $x = 0$—that is, at $x = \pi/2 + n\pi$, where n is an integer. Similarly,

$$\cot x = \frac{\cos x}{\sin x} \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

have vertical asymptotes where sin $x = 0$—that is, at $x = n\pi$, where n is an integer.

Graphs of the Reciprocal Functions

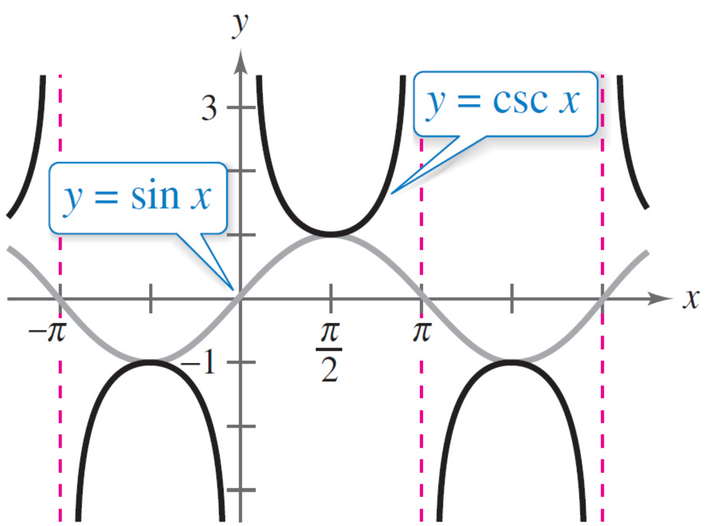
To sketch the graph of a secant or cosecant function, you should first make a sketch of its reciprocal function.

For instance, to sketch the graph of $y = \csc x$, first sketch the graph of $y = \sin x$.

Then take reciprocals of the y-coordinates to obtain points on the graph of $y = \csc x$.

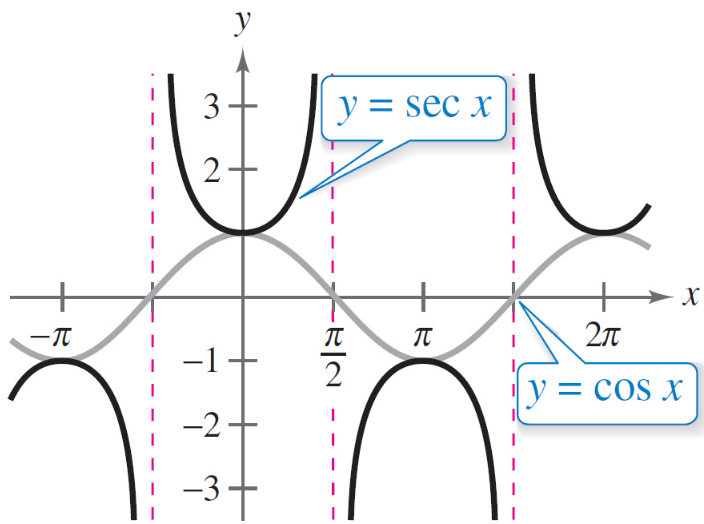
Graphs of the Reciprocal Functions

You can use this procedure to obtain the graphs shown below.



Period: 2π
Domain: all $x \neq n\pi$
Range: $(-\infty, -1] \cup [1, \infty)$
Vertical asymptotes: $x = n\pi$
Symmetry: origin

Graphs of the Reciprocal Functions



Period: 2π

Domain: all $x \neq \frac{\pi}{2} + n\pi$

Range: $(-\infty, -1] \cup [1, \infty)$

Vertical asymptotes: $x = \frac{\pi}{2} + n\pi$

Symmetry: y-axis

In comparing the graphs of the cosecant and secant functions with those of the sine and cosine functions, respectively, note that the “hills” and “valleys” are interchanged.

Graphs of the Reciprocal Functions

For instance, a hill (or maximum point) on the sine curve corresponds to a valley (a relative minimum) on the cosecant curve, and a valley (or minimum point) on the sine curve corresponds to a hill (a relative maximum) on the cosecant curve, as shown in Figure 1.47.

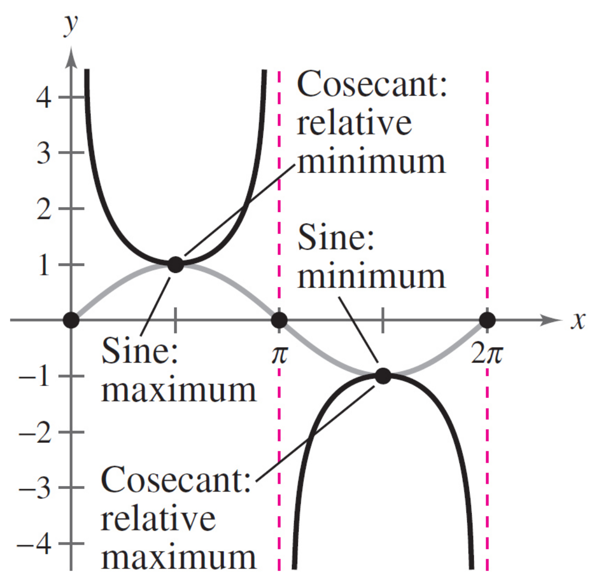


Figure 1.47

Graphs of the Reciprocal Functions

Additionally, *x*-intercepts of the sine and cosine functions become vertical asymptotes of the cosecant and secant functions, respectively (see Figure 1.47).

Example 4 – *Sketching the Graph of a Cosecant Function*

Sketch the graph of $y = 2 \csc\left(x + \frac{\pi}{4}\right)$.

Solution:

Begin by sketching the graph of

$$y = 2 \sin\left(x + \frac{\pi}{4}\right).$$

For this function, the amplitude is 2 and the period is 2π.

Example 4 – *Solution*

cont'd

By solving the equations

$$x + \frac{\pi}{4} = 0 \qquad \text{and} \qquad x + \frac{\pi}{4} = 2\pi$$

$$x = -\frac{\pi}{4} \qquad\qquad x = \frac{7\pi}{4}$$

you can see that one cycle of the sine function corresponds to the interval from $x = -\pi/4$ to $x = 7\pi/4$.

Example 4 – *Solution*

cont'd

The graph of this sine function is represented by the gray curve in Figure 1.48.

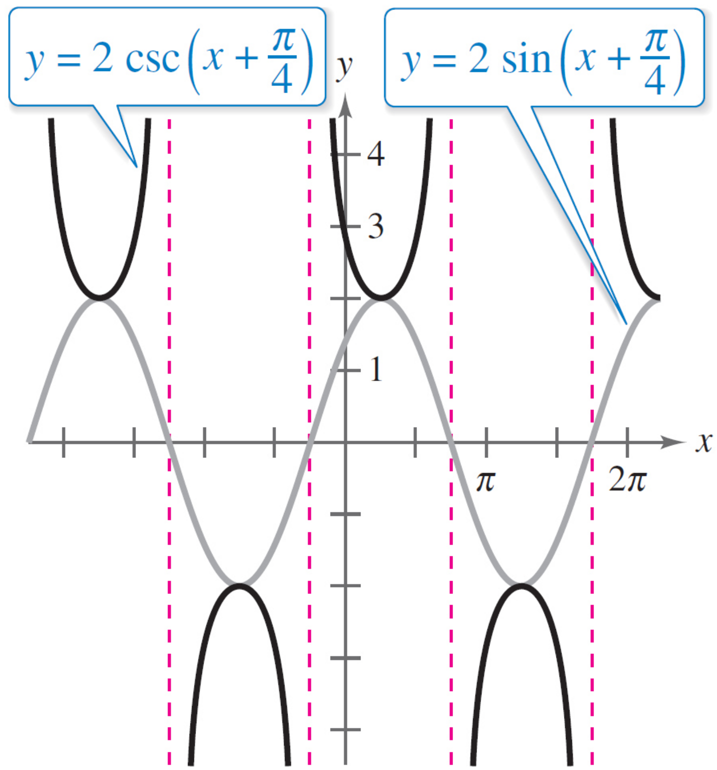


Figure 1.48

Example 4 – *Solution*

cont'd

Because the sine function is zero at the midpoint and endpoints of this interval, the corresponding cosecant function

$$y = 2\csc\left(x + \frac{\pi}{4}\right)$$

$$= 2\left(\frac{1}{\sin[x + (\pi/4)]}\right)$$

has vertical asymptotes at $x = -\pi/4$, $x = 3\pi/4$, $x = 7\pi/4$ and so on.

The graph of the cosecant function is represented by the black curve in Figure 1.48.

Damped Trigonometric Graphs

Damped Trigonometric Graphs

You can graph a *product* of two functions using properties of the individual functions.

For instance, consider the function

$$f(x) = x \sin x$$

as the product of the functions $y = x$ and $y = \sin x$.

Using properties of absolute value and the fact that $|\sin x| \leq 1$, you have $0 \leq |x||\sin x| \leq |x|$.

Damped Trigonometric Graphs

Consequently,

$$-|x| \le x \sin x \le |x|$$

which means that the graph of $f(x) = x \sin x$ lies between the lines $y = -x$ and $y = x$.

Furthermore, because

$$f(x) = x \sin x = \pm x \qquad \text{at} \qquad x = \frac{\pi}{2} + n\pi$$

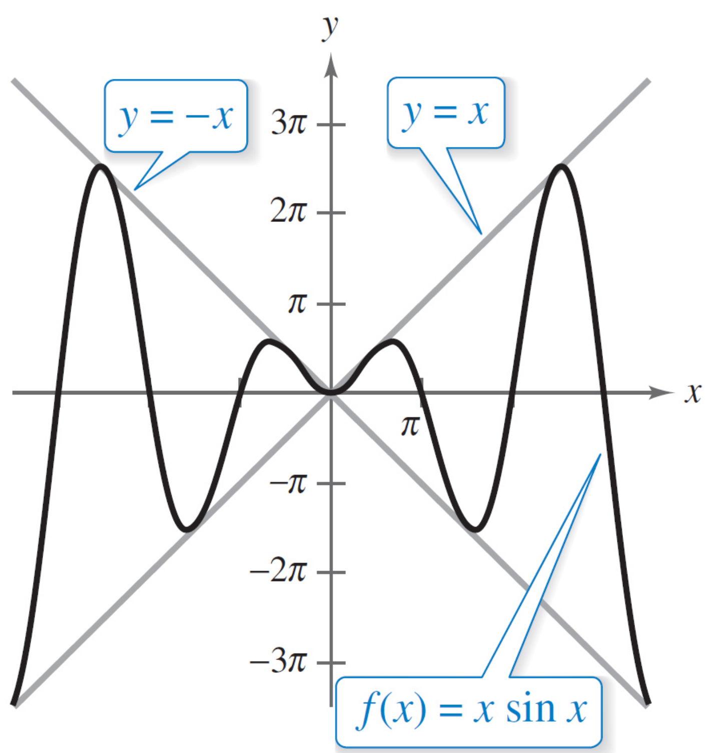
and

$$f(x) = x \sin x = 0 \qquad \text{at} \qquad x = n\pi$$

where n is an integer, the graph of f touches the line $y = -x$ or the line $y = x$ at $x = \pi/2 + n\pi$ and has x-intercepts at $x = n\pi$.

Damped Trigonometric Graphs

A sketch of f is shown below. In the function $f(x) = x \sin x$, the factor x is called the **damping factor**.



Example 6 – *Damped Sine Wave*

Sketch the graph of $f(x) = x^2 \sin 3x$.

Solution:

Consider $f(x)$ as the product of the two functions

$y = x^2$ and $y = \sin 3x$

each of which has the set of real numbers as its domain.

For any real number x, you know that $x^2 \geq 0$ and $|\sin 3x| \leq 1$.

So, $x^2 |\sin 3x| \leq x^2$, which means that $-x^2 \leq x^2 \sin 3x \leq x^2$.

Example 6 – *Solution*

cont'd

Furthermore, because

$$f(x) = x^2 \sin 3x = \pm x^2 \qquad \text{at} \qquad x = \frac{\pi}{6} + \frac{n\pi}{3}$$

and

$$f(x) = x^2 \sin 3x = 0 \qquad \text{at} \qquad x = \frac{n\pi}{3}$$

the graph of f touches the curves $y = -x^2$ and $y = x^2$ at $x = \pi/6 + n\pi/3$ and has intercepts at $x = n\pi/3$.

Example 6 – *Solution*

cont'd

A sketch of *f* is shown in Figure 1.50.

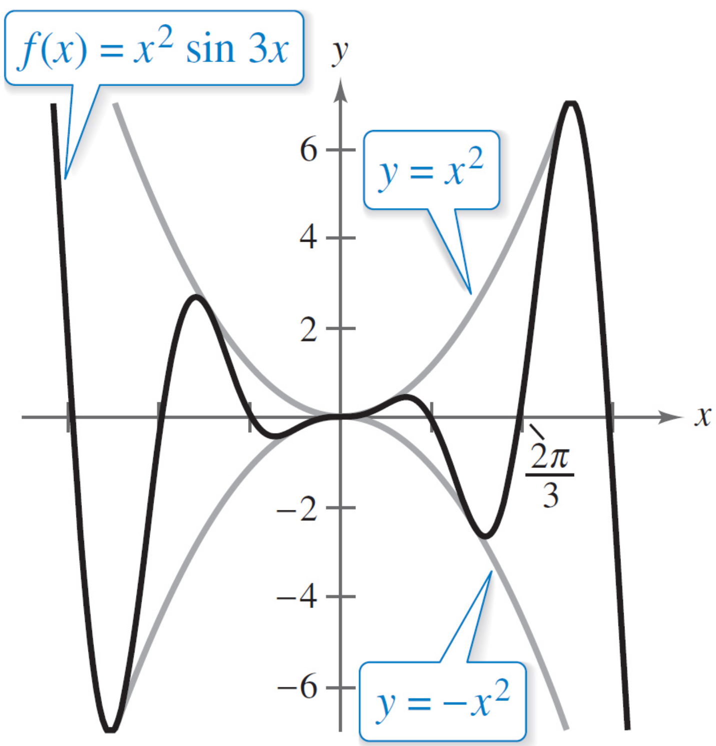
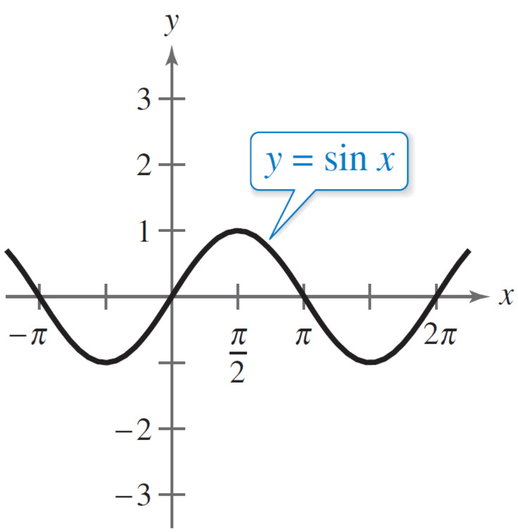


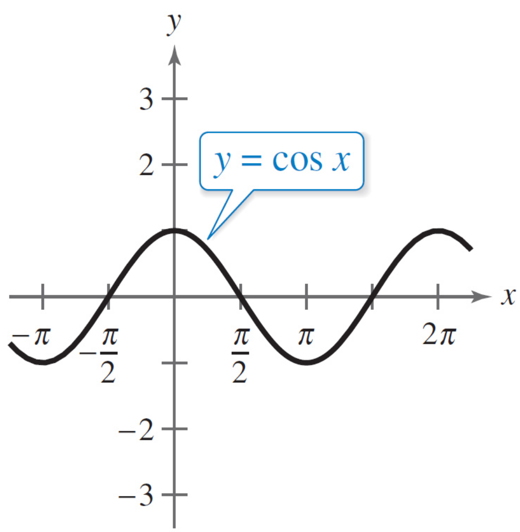
Figure 1.50

Damped Trigonometric Graphs

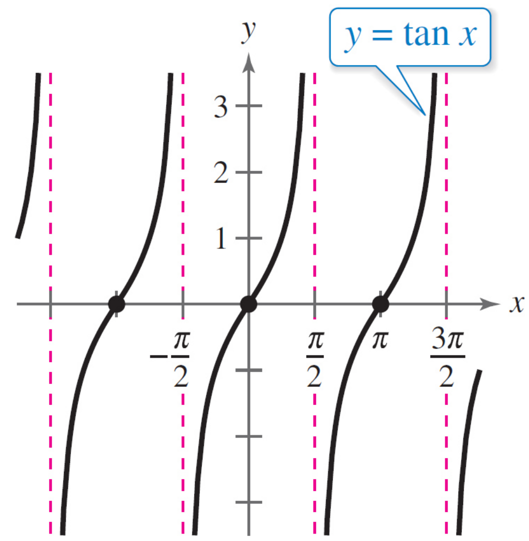
Below is a summary of the characteristics of the six basic trigonometric functions.



Domain: $(-\infty, \infty)$
Range: $[-1, 1]$
Period: 2π



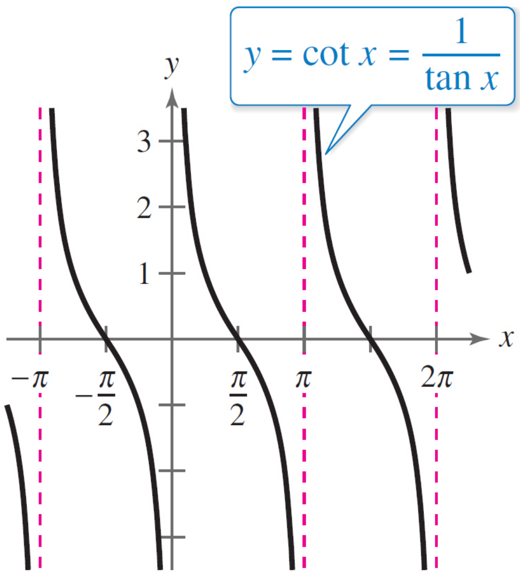
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$
Period: 2π



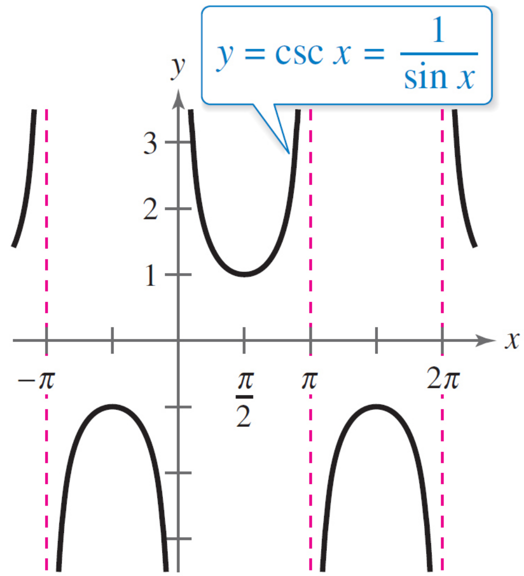
Domain: all $x \neq \frac{\pi}{2} + n\pi$
Range: $(-\infty, \infty)$
Period: π

Damped Trigonometric Graphs

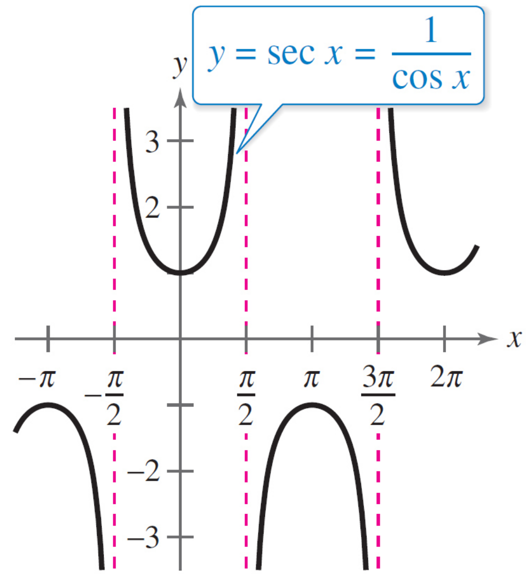
cont'd



Domain: all $x \neq n\pi$
Range: $(-\infty, \infty)$
Period: π



Domain: all $x \neq n\pi$
Range:
$(-\infty, -1] \cup [1, \infty)$
Period: 2π



Domain: all $x \neq \frac{\pi}{2} + n\pi$
Range:
$(-\infty, -1] \cup [1, \infty)$
Period: 2π