1 Trigonometry

1.5 Graphs of Sine and Cosine Functions

Objectives

- Sketch the graphs of basic sine and cosine functions.
- Use amplitude and period to help sketch the graphs of sine and cosine functions.
- Sketch translations of the graphs of sine and cosine functions.
- Use sine and cosine functions to model real-life data.

Basic Sine and Cosine Curves

Basic Sine and Cosine Curves

In this section, you will study techniques for sketching the graphs of the sine and cosine functions. The graph of the sine function is a **sine curve**.

Basic Sine and Cosine Curves

In Figure 1.36, the black portion of the graph represents one period of the function and is called **one cycle** of the sine curve.

The gray portion of the graph indicates that the basic sine curve repeats indefinitely to the left and right.

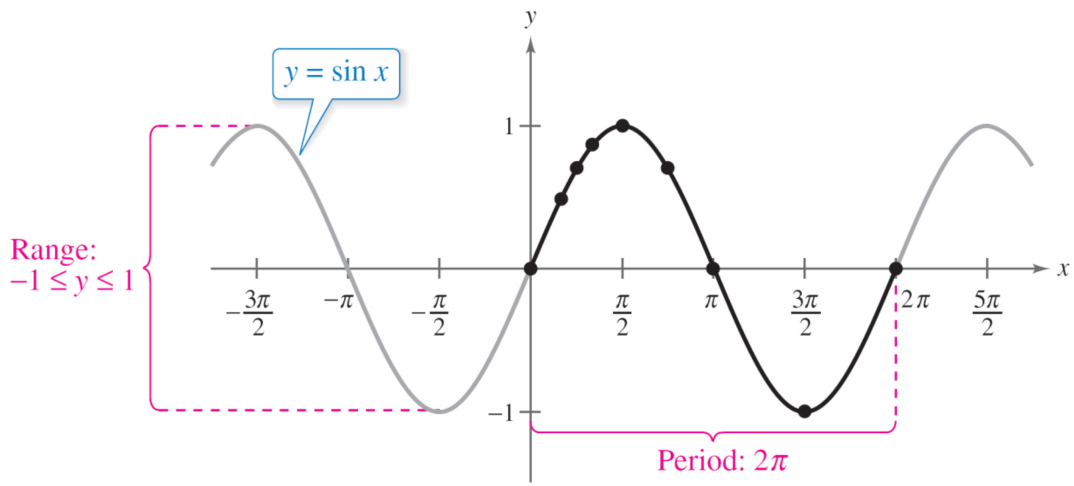


Figure 1.36

Basic Sine and Cosine Curves

The graph of the cosine function is shown in Figure 1.37.

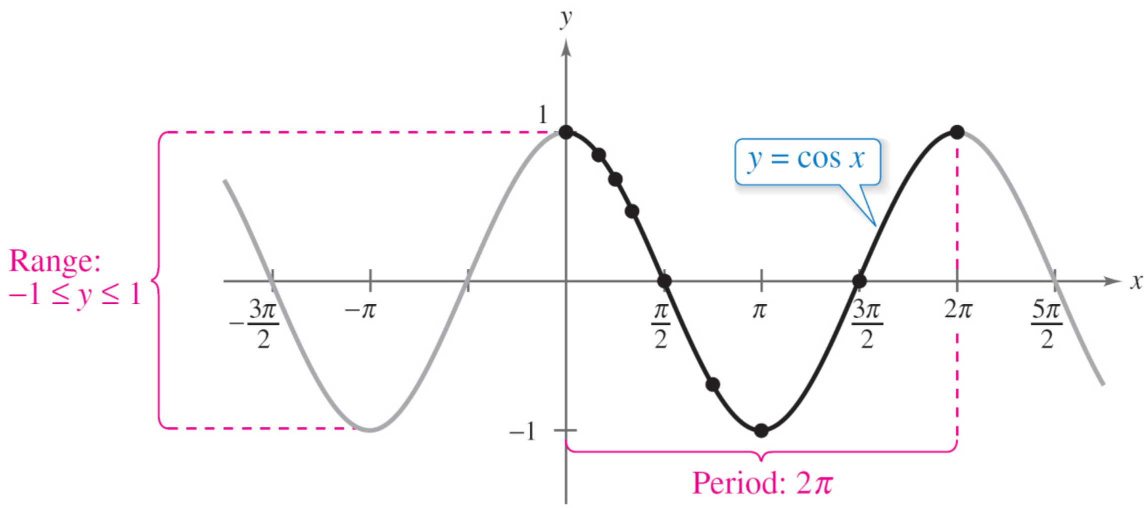


Figure 1.37

Basic Sine and Cosine Curves

We know that the domain of the sine and cosine functions is the set of all real numbers.

Moreover, the range of each function is the interval [–1, 1], and each function has a period of 2π. Do you see how this information is consistent with the basic graphs shown in Figures 1.36 and 1.37?

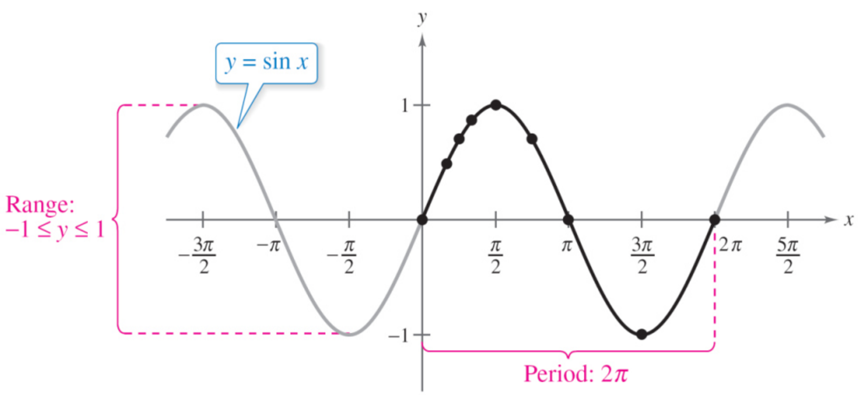


Figure 1.36

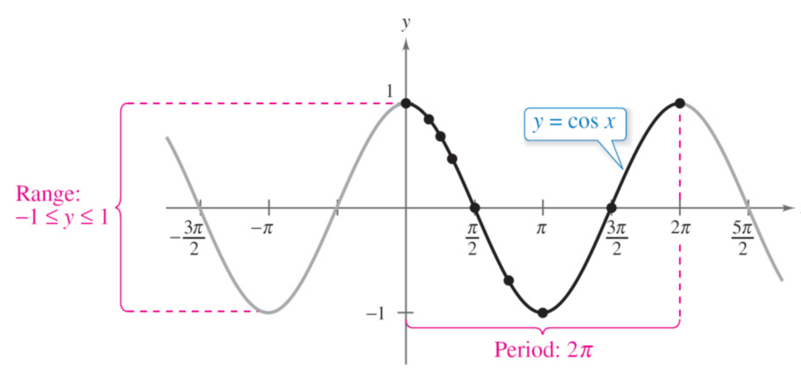


Figure 1.37

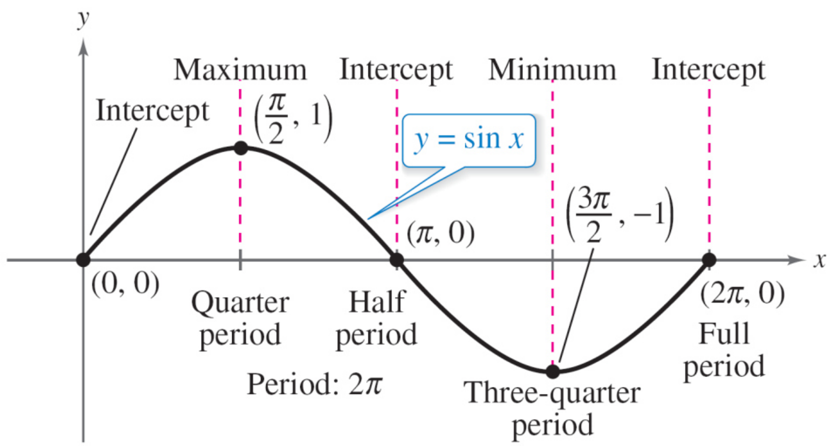
Basic Sine and Cosine Curves

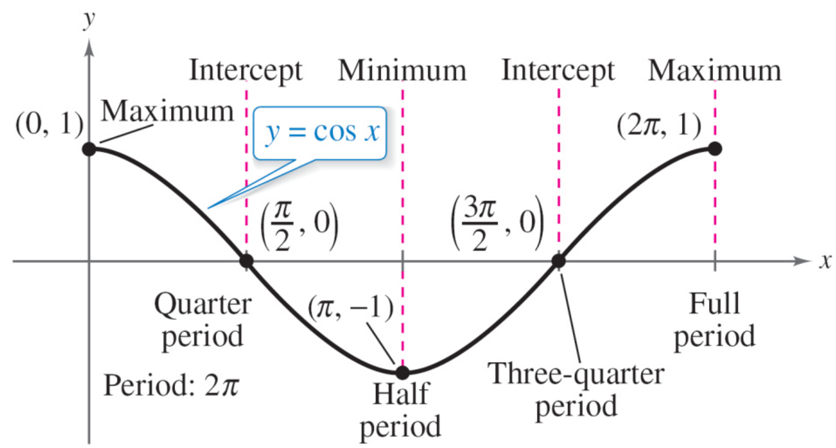
Note in Figures 1.36 and 1.37 that the sine curve is symmetric with respect to the *origin*, whereas the cosine curve is symmetric with respect to the *y-axis*.

These properties of symmetry follow from the fact that the sine function is odd and the cosine function is even.

Basic Sine and Cosine Curves

To sketch the graphs of the basic sine and cosine functions by hand, it helps to note five **key points** in one period of each graph: the *intercepts*, *maximum points*, and *minimum points* (see below).





Example 1 – *Using Key Points to Sketch a Sine Curve*

Sketch the graph of $y = 2 \sin x$ on the interval $[-\pi, 4\pi]$.

Solution:

Note that

$$y = 2 \sin x = 2(\sin x)$$

indicates that the y-values for the key points will have twice the magnitude of those on the graph of $y = \sin x$.

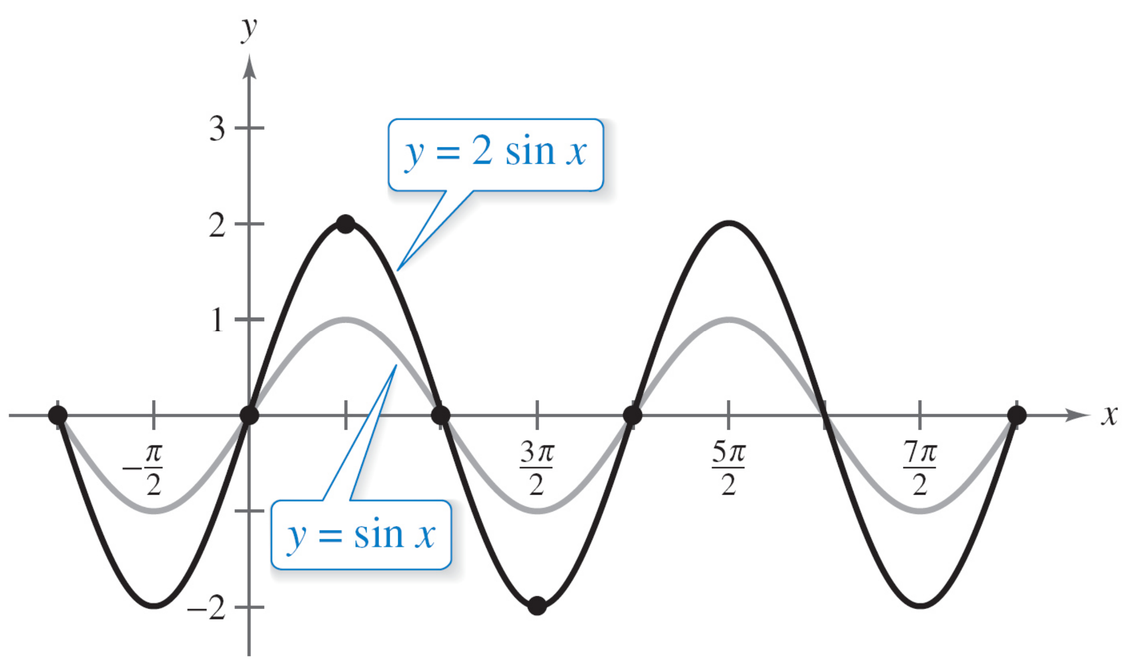
Divide the period 2π into four equal parts to get the key points

Intercept	**Maximum**	**Intercept**	**Minimum**		**Intercept**
$(0, 0)$,	$\left(\frac{\pi}{2}, 2\right)$,	$(\pi, 0)$,	$\left(\frac{3\pi}{2}, -2\right)$,	and	$(2\pi, 0)$.

Example 1 – *Solution*

cont'd

By connecting these key points with a smooth curve and extending the curve in both directions over the interval $[-\pi, 4\pi]$, you obtain the graph shown below.



Amplitude and Period

Amplitude and Period

In the rest of this section, you will study the graphic effect of each of the constants a, b, c, and d in equations of the forms

$$y = d + a\sin(bx - c)$$

and

$$y = d + a\cos(bx - c).$$

The constant factor a in $y = a \sin x$ acts as a *scaling factor*—a *vertical stretch* or *vertical shrink* of the basic sine curve. When $|a| > 1$, the basic sine curve is stretched, and when $|a| < 1$, the basic sine curve is shrunk.

Amplitude and Period

The result is that the graph of $y = a \sin x$ ranges between $-a$ and a instead of between -1 and 1.

The absolute value of a is the **amplitude** of the function $y = a \sin x$. The range of the function $y = a \sin x$ for $a > 0$ is $-a \leq y \leq a$.

Definition of Amplitude of Sine and Cosine Curves

The **amplitude** of $y = a \sin x$ and $y = a \cos x$ represents half the distance between the maximum and minimum values of the function and is given by

$$\text{Amplitude} = |a|.$$

Amplitude and Period

The graph of $y = -f(x)$ is a **reflection** in the x-axis of the graph of $y = f(x)$.

For instance, the graph of $y = -3 \cos x$ is a reflection of the graph of $y = 3 \cos x$, as shown in Figure 1.39.

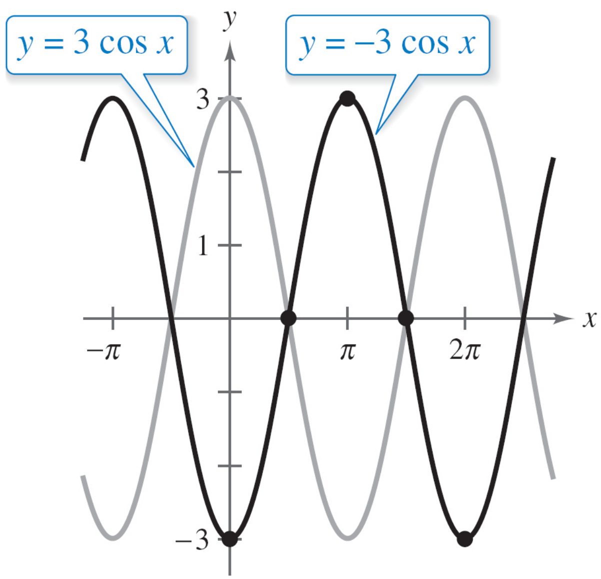


Figure 1.39

Amplitude and Period

Because $y = a \sin x$ completes one cycle from $x = 0$ to $x = 2\pi$, it follows that $y = a \sin bx$ completes one cycle from $x = 0$ to $x = 2\pi/b$, where b is a positive real number.

Period of Sine and Cosine Functions

Let b be a positive real number. The **period** of $y = a \sin bx$ and $y = a \cos bx$ is given by

$$\text{Period} = \frac{2\pi}{b}.$$

Note that when $0 < b < 1$, the period of $y = a \sin bx$ is greater than 2π and represents a *horizontal stretching* of the graph of $y = a \sin x$.

Amplitude and Period

Similarly, when $b > 1$, the period of $y = a \sin bx$ is less than 2π and represents a *horizontal shrinking* of the graph of $y = a \sin x$.

When b is negative, the identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ are used to rewrite the function.

Example 3 – *Scaling: Horizontal Stretching*

Sketch the graph of $y = \sin \dfrac{x}{2}$.

Solution:

The amplitude is 1. Moreover, because $b = \frac{1}{2}$, the period is

$$\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} \qquad \text{Substitute for } b.$$

$$= 4\pi.$$

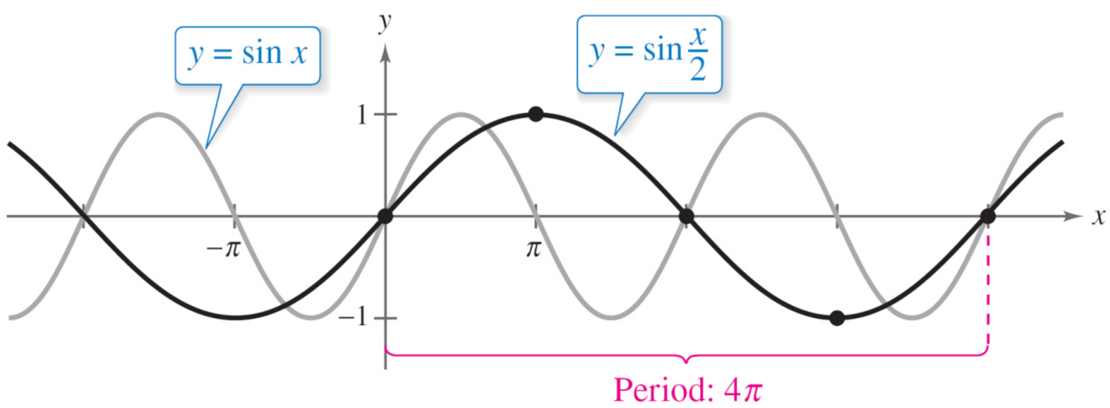
Example 3 – *Solution*

cont'd

Now, divide the period-interval [0, 4π] into four equal parts with the values π, 2π, and 3π to obtain the key points

Intercept	**Maximum**	**Intercept**	**Minimum**		**Intercept**
(0, 0),	(π, 1),	(2π, 0),	(3π, –1),	and	(4π, 0).

The graph is shown below.



Translations of Sine and Cosine Curves

Translations of Sine and Cosine Curves

The constant c in the general equations

$$y = a\sin(bx - c) \qquad \text{and} \qquad y = a\cos(bx - c)$$

creates *horizontal translations* (shifts) of the basic sine and cosine curves.

Comparing $y = a \sin bx$ with $y = a\sin(bx - c)$, you find that the graph of $y = a\sin(bx - c)$ completes one cycle from $bx - c = 0$ to $bx - c = 2\pi$.

Translations of Sine and Cosine Curves

By solving for *x*, you can find the interval for one cycle to be

$$\underbrace{\frac{c}{b}}^{\text{Left endpoint}} \le x \le \overbrace{\frac{c}{b} + \underbrace{\frac{2\pi}{b}}_{\text{Period}}}^{\text{Right endpoint}}.$$

This implies that the period of $y = a \sin(bx - c)$ is $2\pi/b$, and the graph of $y = a \sin bx$ is shifted by an amount c/b.

The number c/b is the **phase shift**.

Translations of Sine and Cosine Curves

Graphs of Sine and Cosine Functions

The graphs of $y = a \sin(bx - c)$ and $y = a \cos(bx - c)$ have the following characteristics. (Assume $b > 0$.)

$$\text{Amplitude} = |a| \qquad \text{Period} = \frac{2\pi}{b}$$

The left and right endpoints of a one-cycle interval can be determined by solving the equations $bx - c = 0$ and $bx - c = 2\pi$.

Example 5 – *Horizontal Translation*

Sketch the graph of

$$y = -3\cos(2\pi x + 4\pi).$$

Solution:

The amplitude is 3 and the period is $2\pi/2\pi = 1$.

By solving the equations

$$2\pi x + 4\pi = 0$$

$$2\pi x = -4\pi$$

$$x = -2$$

and

$$2\pi x + 4\pi = 2\pi$$

Example 5 – *Solution*

cont'd

$$2\pi x = -2\pi$$

$$x = -1$$

you see that the interval [–2, –1] corresponds to one cycle of the graph.

Dividing this interval into four equal parts produces the key points

Minimum	Intercept	Maximum	Intercept	Minimum
$(-2, -3)$,	$\left(-\frac{7}{4}, 0\right)$,	$\left(-\frac{3}{2}, 3\right)$,	$\left(-\frac{5}{4}, 0\right)$,	and $(-1, -3)$.

Example 5 – *Solution*

cont'd

The graph is shown in Figure 1.40

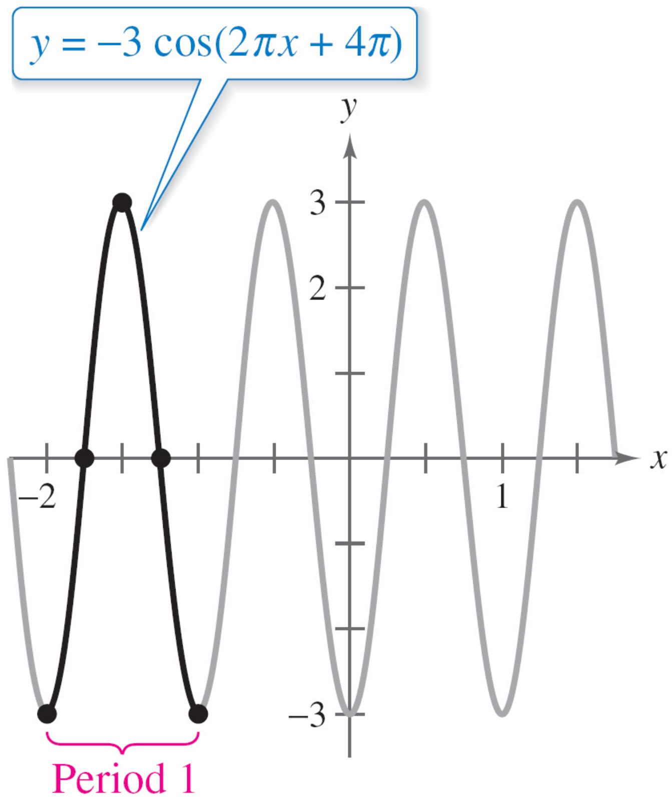


Figure 1.40

Translations of Sine and Cosine Curves

The final type of transformation is the *vertical translation* caused by the constant d in the equations

$$y = d + a\sin(bx - c)$$

and

$$y = d + a\cos(bx - c).$$

The shift is d units up for $d > 0$ and d units down for $d < 0$.

In other words, the graph oscillates about the horizontal line $y = d$ instead of about the x-axis.

Mathematical Modeling

Example 7 – *Finding a Trigonometric Model*

The table shows the depths (in feet) of the water at the end of a dock at various times during the morning, where $t = 0$ corresponds to midnight.

Time, t	Depth, y
0	3.4
2	8.7
4	11.3
6	9.1
8	3.8
10	0.1
12	1.2

Example 7 – *Finding a Trigonometric Model* cont'd

a. Use a trigonometric function to model the data.

b. Find the depths at 9 A.M. and 3 P.M.

c. A boat needs at least 10 feet of water to moor at the dock. During what times in the afternoon can it safely dock?

Example 7(a) – *Solution*

Begin by graphing the data, as shown in Figure 1.42.

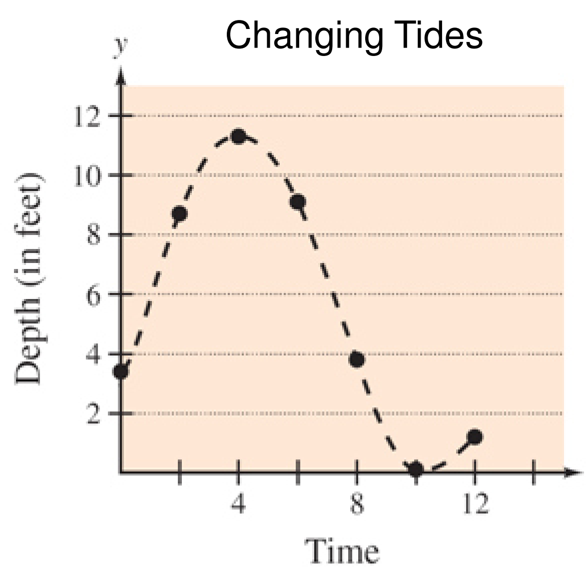


Figure 1.42

You can use either a sine or a cosine model. Suppose you use a cosine model of the form $y = a\cos(bt - c) + d$.

Example 7(a) – *Solution*

cont'd

The difference between the maximum value and the minimum value is twice the amplitude of the function. So, the amplitude is

$$a = \frac{1}{2}[(\text{maximum depth}) - (\text{minimum depth})]$$

$$= \frac{1}{2}(11.3 - 0.1)$$

$$= 5.6.$$

The cosine function completes one half of a cycle between the times at which the maximum and minimum depths occur. So, the period p is

$$p = 2[(\text{time of min. depth}) - (\text{time of max. depth})]$$

Example 7(a) – *Solution*

cont'd

$$= 2(10 - 4)$$

$$= 12$$

which implies that

$$b = 2\pi/p$$

$$\approx 0.524.$$

Because high tide occurs 4 hours after midnight, consider the left endpoint to be $c/b = 4$, so $c \approx 2.094$.

Example 7(a) – *Solution*

cont'd

Moreover, because the average depth is $\frac{1}{2}(11.3 + 0.1) = 5.7$, it follows that $d = 5.7$.

So, you can model the depth with the function

$$y = 5.6 \cos(0.524t - 2.094) + 5.7.$$

Example 7(b) – *Solution*

cont'd

The depths at 9 A.M. and 3 P.M. are as follows.

$$y = 5.6\cos(0.524 \cdot 9 - 2.094) + 5.7$$

$$\approx 0.84 \text{ foot} \qquad \text{9 A.M.}$$

$$y = 5.6\cos(0.524 \cdot 15 - 2.094) + 5.7$$

$$\approx 10.57 \text{ foot} \qquad \text{3 P.M.}$$

Example 7(c) – *Solution*

cont'd

Using a graphing utility, graph the model with the line $y = 10$.

Using the *intersect* feature, you can determine that the depth is at least 10 feet between 2:42 P.M. ($t \approx 14.7$) and 5:18 P.M. ($t \approx 17.3$), as shown in Figure 1.43.

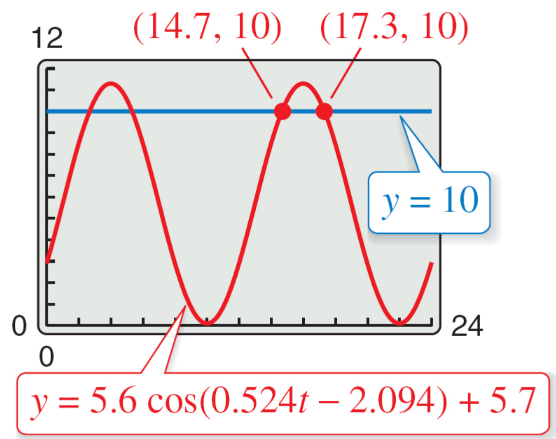


Figure 1.43