1 Trigonometry

1.4

Trigonometric Functions of Any Angle

Objectives

- Evaluate trigonometric functions of any angle.
- Find reference angles.
- Evaluate trigonometric functions of real numbers.

Introduction

Introduction

The definitions of trigonometric functions were restricted to acute angles. In this section, the definitions are extended to cover *any* angle. When θ is an *acute* angle, the definitions here coincide with those given in the preceding section.

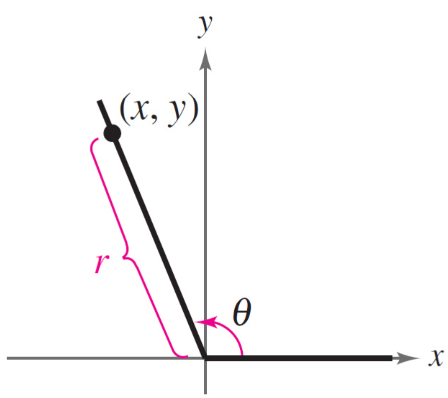
Definitions of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$$\sin \theta = \frac{y}{r} \qquad \cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}, \quad x \neq 0 \qquad \cot \theta = \frac{x}{y}, \quad y \neq 0$$

$$\sec \theta = \frac{r}{x}, \quad x \neq 0 \qquad \csc \theta = \frac{r}{y}, \quad y \neq 0$$



Introduction

Because $r = \sqrt{x^2 + y^2}$ *cannot* be zero, it follows that the sine and cosine functions are defined for any real value of θ.

However, when $x = 0$, the tangent and secant of θ are undefined.

For example, the tangent of 90° is undefined. Similarly, when $y = 0$, the cotangent and cosecant of θ are undefined.

Example 1 – *Evaluating Trigonometric Functions*

Let (–3, 4) be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.

Solution:

Referring to Figure 1.27, you can see that $x = -3$, $y = 4$, and

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
&= \sqrt{(-3)^2 + 4^2} \\
&= \sqrt{25} \\
&= 5.
\end{aligned}
$$

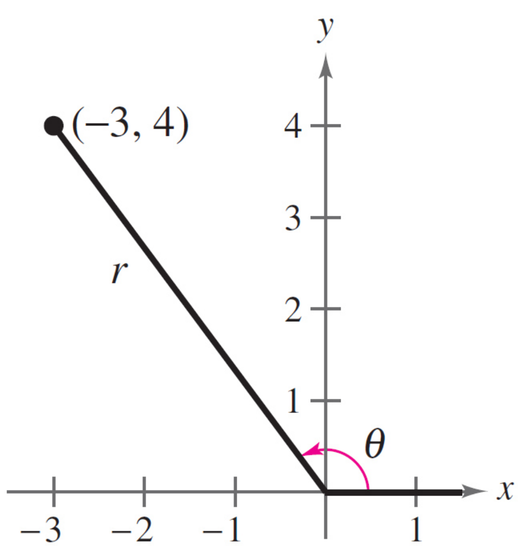


Figure 1.27

Example 1 – *Solution*

cont'd

So, you have the following.

$$\sin\,\theta = \frac{y}{r} = \frac{4}{5}$$

$$\cos\,\theta = \frac{x}{r} = -\frac{3}{5}$$

$$\tan\,\theta = \frac{y}{x} = -\frac{4}{3}$$

Introduction

The *signs* of the trigonometric functions in the four quadrants can be determined from the definitions of the functions.

For instance, because $\cos\theta = x/r$, it follows that $\cos\theta$ is positive wherever $x > 0$, which is in Quadrants I and IV. (Remember, r is always positive.)

Introduction

In a similar manner, you can verify the results shown in Figure 1.28.

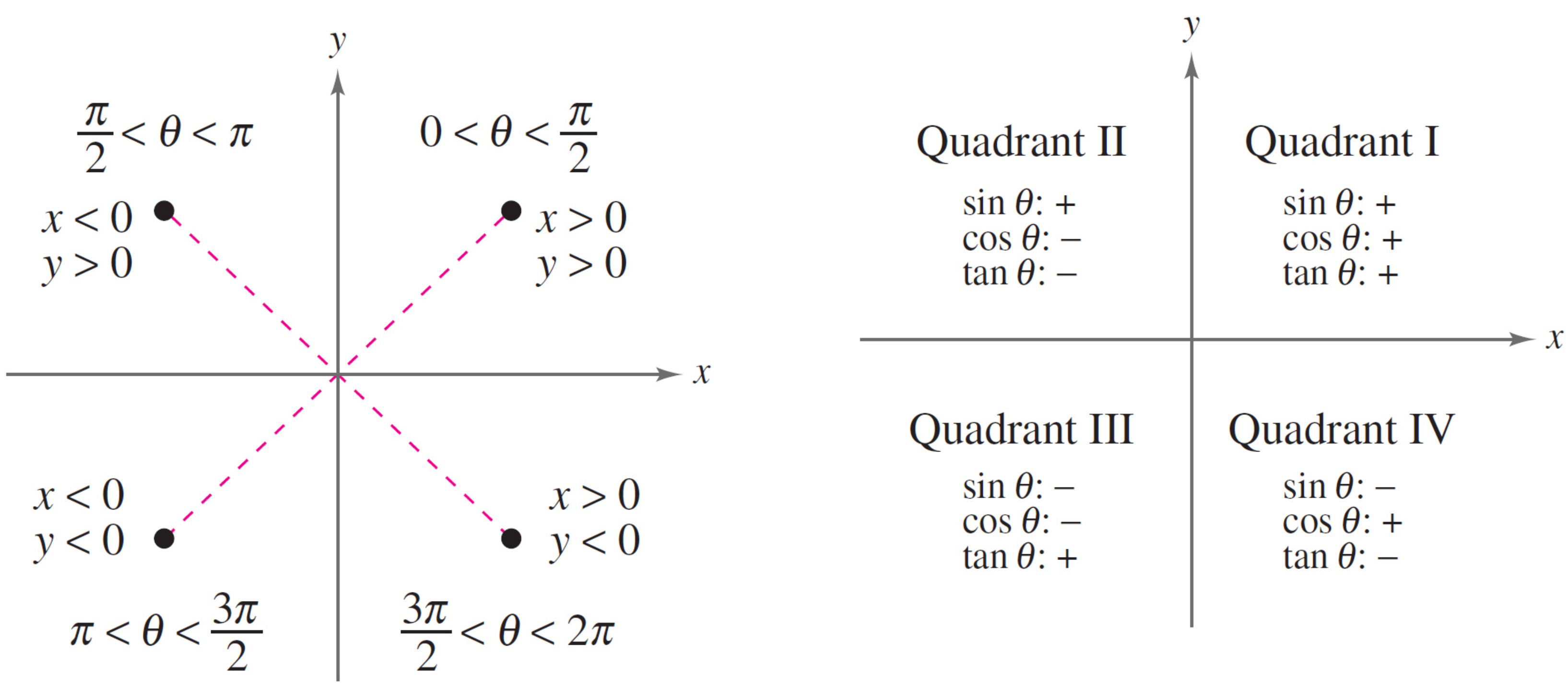


Figure 1.28

Reference Angles

Reference Angles

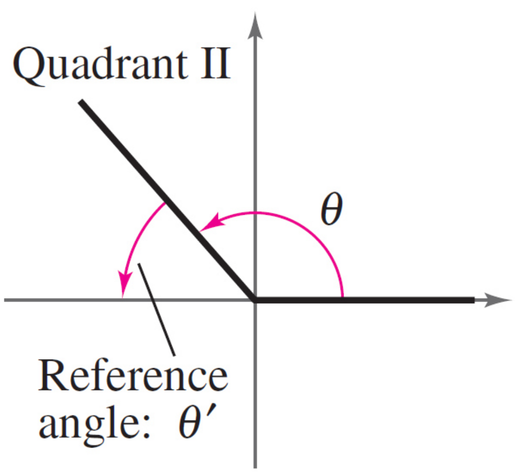
The values of the trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding acute angles called **reference angles.**

Definition of Reference Angle

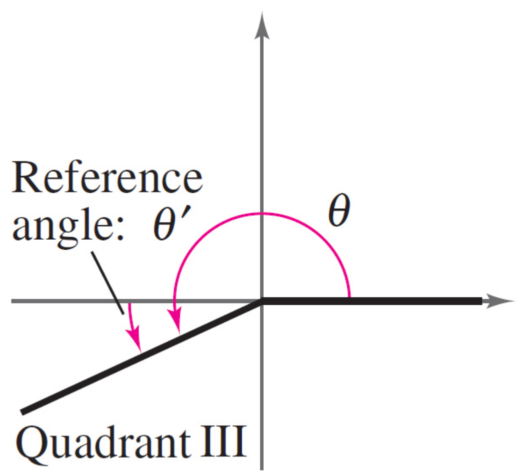
Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Reference Angles

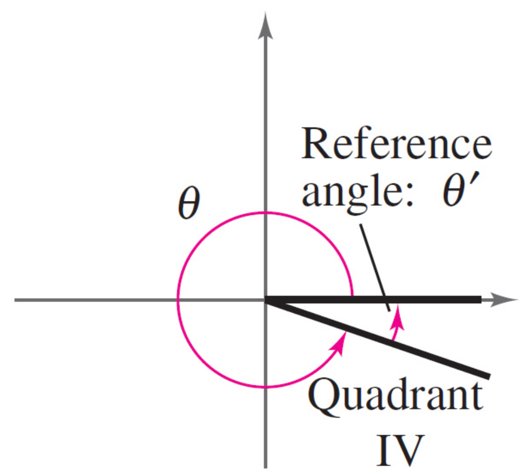
The reference angles for θ in Quadrants II, III, and IV are shown below.



$\theta' = \pi - \theta$ (radians)
$\theta' = 180° - \theta$ (degrees)



$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180°$ (degrees)



$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360° - \theta$ (degrees)

Example 4 – *Finding Reference Angles*

Find the reference angle θ'.

a. $\theta = 300°$

b. $\theta = 2.3$

c. $\theta = -135°$

Example 4(a) – *Solution*

Because 300° lies in Quadrant IV, the angle it makes with the x-axis is

$$\theta' = 360° - 300°$$

$$= 60°.$$

Degrees

Figure 1.30 shows the angle $\theta = 300°$ and its reference angle $\theta' = 60°$.

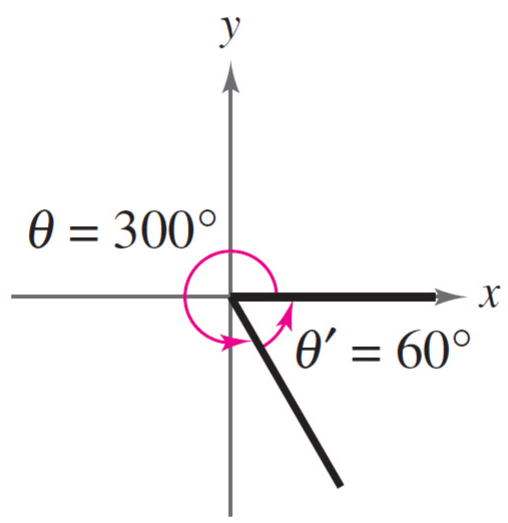


Figure 1.30

Example 4(b) – *Solution*

cont'd

Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is

$$\theta' = \pi - 2.3$$

$$\approx 0.8416.$$

Radians

Figure 1.31 shows the angle $\theta = 2.3$ and its reference angle $\theta' = \pi - 2.3$.

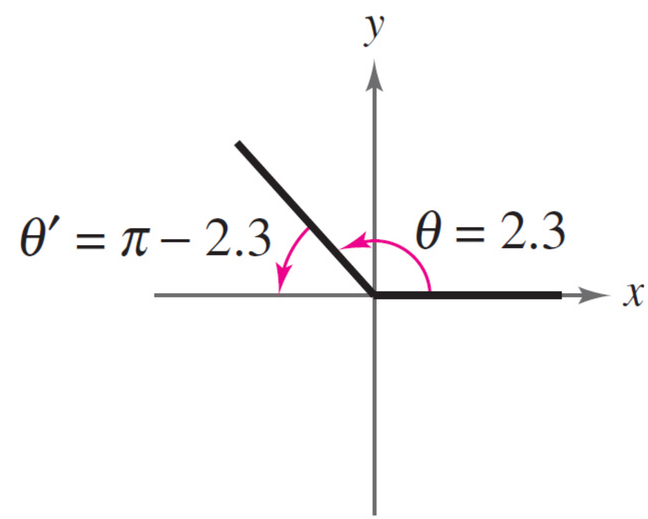


Figure 1.31

Example 4(c) – *Solution*

cont'd

First, determine that –135° is coterminal with 225°, which lies in Quadrant III. So, the reference angle is

$$\theta' = 225° - 180°$$

$$= 45°.$$

Degrees

Figure 1.32 shows the angle $\theta = -135°$ and its reference angle $\theta' = 45°$.

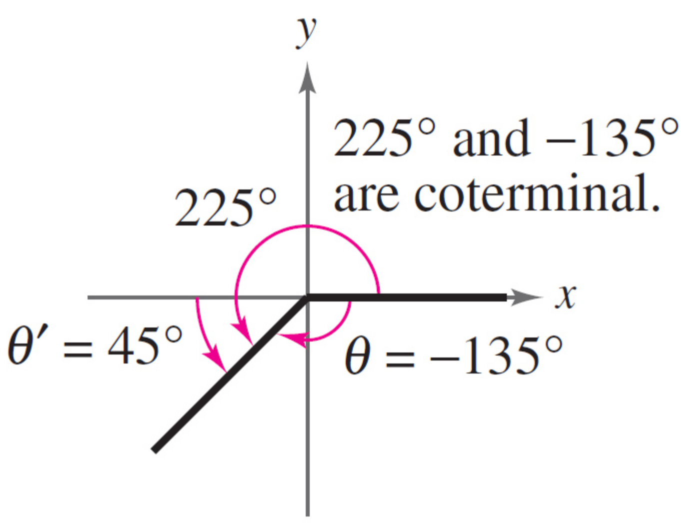
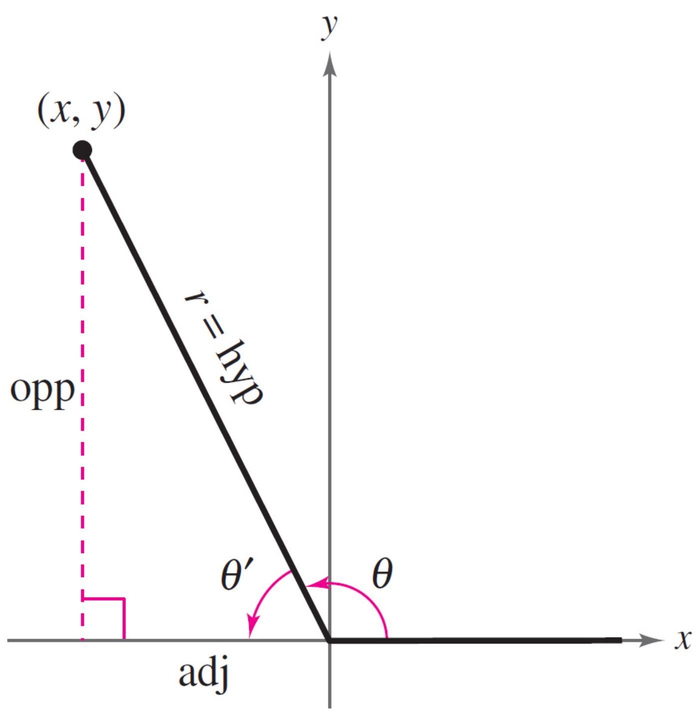


Figure 1.32

Trigonometric Functions of Real Numbers

Trigonometric Functions of Real Numbers

To see how a reference angle is used to evaluate a trigonometric function, consider the point (x, y) on the terminal side of θ, as shown in figure below.



opp $= |y|$, adj $= |x|$

Trigonometric Functions of Real Numbers

By definition, you know that

$$\sin\,\theta = \frac{y}{r} \qquad \text{and} \qquad \tan\,\theta = \frac{y}{x}.$$

For the right triangle with acute angle θ' and sides of lengths $|x|$ and $|y|$, you have

$$\sin\theta' = \frac{\text{opp}}{\text{hyp}} = \frac{|y|}{r}$$

and

$$\tan\theta' = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|}.$$

Trigonometric Functions of Real Numbers

So, it follows that sin θ and sin θ' are equal, *except possibly in sign*. The same is true for tan θ and tan θ' *and* for the other four trigonometric functions.

In all cases, the quadrant in which θ lies determines the sign of the function value.

Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle θ:

1. Determine the function value of the associated reference angle θ'.
2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

Trigonometric Functions of Real Numbers

You can greatly extend the scope of *exact* trigonometric values.

For instance, knowing the function values of 30° means that you know the function values of all angles for which 30° is a reference angle.

Trigonometric Functions of Real Numbers

For convenience, the table below shows the exact values of the sine, cosine, and tangent functions of special angles and quadrant angles.

Trigonometric Values of Common Angles

θ (degrees)	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\ \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\ \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\ \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.

Example 5 – *Using Reference Angles*

Evaluate each trigonometric function.

a. $\cos \dfrac{4\pi}{3}$

b. $\tan(-210°)$

c. $\csc \dfrac{11\pi}{4}$

Example 5(a) – *Solution*

Because $\theta = 4\pi/3$ lies in Quadrant III, the reference angle is

$$\theta' = \frac{4\pi}{3} - \pi$$

$$= \frac{\pi}{3}$$

as shown in Figure 1.33.

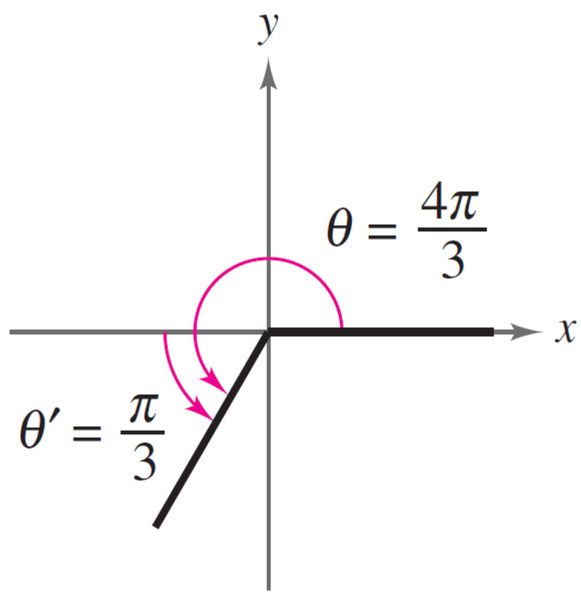


Figure 1.33

Moreover, the cosine is negative in Quadrant III, so

$$\cos\frac{4\pi}{3} = (-)\cos\frac{\pi}{3}$$

$$= -\frac{1}{2}.$$

Example 5(b) – *Solution*

cont'd

Because –210° + 360° = 150°, it follows that –210° is coterminal with the second-quadrant angle 150°.

So, the reference angle is θ' = 180° – 150° = 30°, as shown in Figure 1.34.

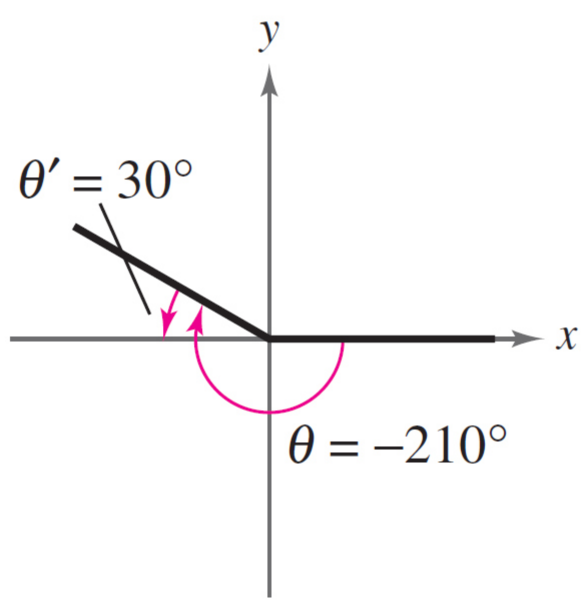


Figure 1.34

Example 5(b) – *Solution*

cont'd

Finally, because the tangent is negative in Quadrant II, you have

$$\tan(-210^\circ) = (-)\tan 30^\circ$$

$$= -\frac{\sqrt{3}}{3}.$$

Example 5(c) – *Solution*

cont'd

Because $(11\pi/4) - 2\pi = 3\pi/4$, it follows that $11\pi/4$ is coterminal with the second-quadrant angle $3\pi/4$.

So, the reference angle is $\theta' = \pi - (3\pi/4) = \pi/4$, as shown in Figure 1.35.

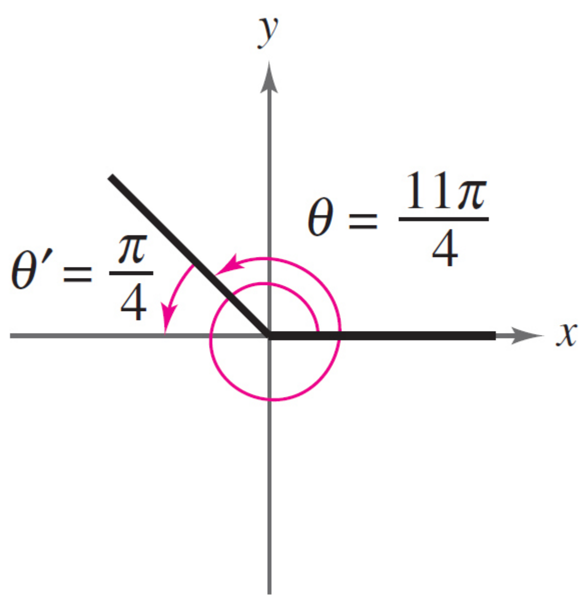


Figure 1.35

Example 5(c) – *Solution*

cont'd

Because the cosecant is positive in Quadrant II, you have

$$\csc \frac{11\pi}{4} = (+) \csc \frac{\pi}{4}$$

$$= \frac{1}{\sin(\pi/4)}$$

$$= \sqrt{2}.$$