1 Trigonometry

1.2 Trigonometric Functions: The Unit Circle

Objectives

- Identify a unit circle and describe its relationship to real numbers.

- Evaluate trigonometric functions using the unit circle.

- Use domain and period to evaluate sine and cosine functions, and use a calculator to evaluate trigonometric functions.

The Unit Circle

The Unit Circle

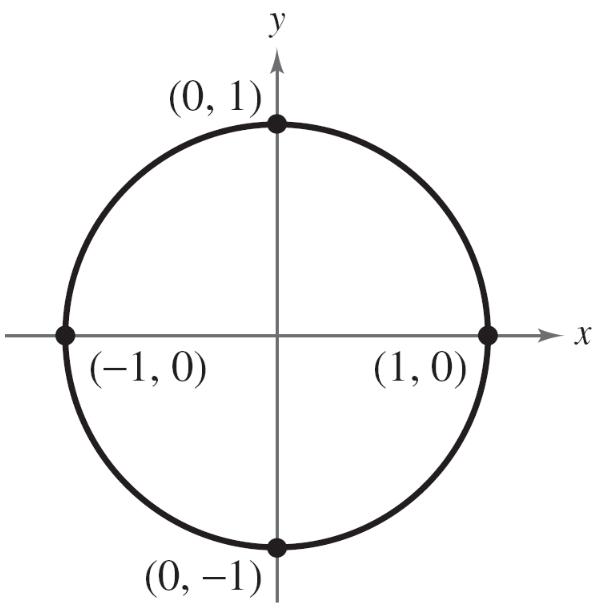
The two historical perspectives of trigonometry incorporate different methods for introducing the trigonometric functions.

One such perspective follows and is based on the unit circle.

Consider the **unit circle** given by

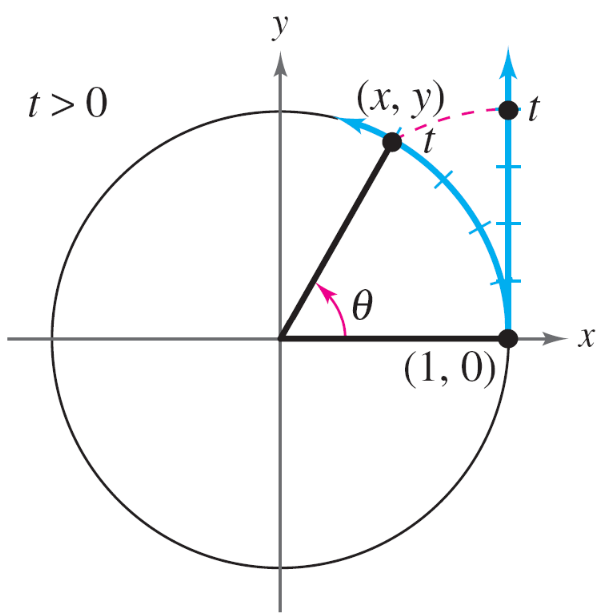
$$x^2 + y^2 = 1 \qquad \text{Unit circle}$$

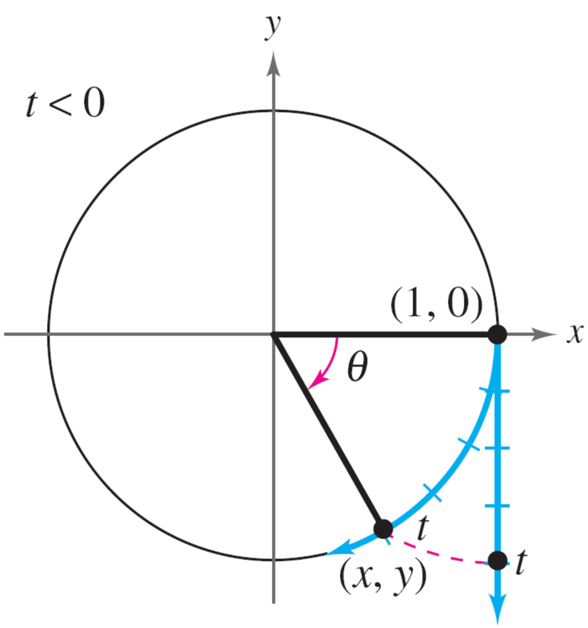
as shown at the right.



The Unit Circle

Imagine wrapping the real number line around this circle, with positive numbers corresponding to a counterclockwise wrapping and negative numbers corresponding to a clockwise wrapping, as shown below.





The Unit Circle

As the real number line wraps around the unit circle, each real number t corresponds to a point (x, y) on the circle.

For example, the real number 0 corresponds to the point (1, 0).

Moreover, because the unit circle has a circumference of 2π, the real number 2π also corresponds to the point (1, 0).

The Unit Circle

In general, each real number t also corresponds to a central angle θ (in standard position) whose radian measure is t.

With this interpretation of t, the arc length formula

$$s = r\theta \qquad\qquad \text{(with } r = 1\text{)}$$

indicates that the real number t is the (directional) length of the arc intercepted by the angle θ, given in radians.

The Trigonometric Functions

The Trigonometric Functions

From the preceding discussion, it follows that the coordinates x and y are two functions of the real variable t.

You can use these coordinates to define the six trigonometric functions of t.

sine   cosecant   cosine   secant   tangent   cotangent

These six functions are normally abbreviated sin, csc, cos, sec, tan, and cot, respectively.

The Trigonometric Functions

Definitions of Trigonometric Functions

Let t be a real number and let (x, y) be the point on the unit circle corresponding to t.

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}, \quad x \neq 0$$

$$\csc t = \frac{1}{y}, \quad y \neq 0 \qquad \sec t = \frac{1}{x}, \quad x \neq 0 \qquad \cot t = \frac{x}{y}, \quad y \neq 0$$

The Trigonometric Functions

In the definitions of the trigonometric functions, note that the tangent and secant are not defined when $x = 0$.

For instance, because $t = \pi/2$ corresponds to $(x, y) = (0, 1)$, it follows that $\tan(\pi/2)$ and $\sec(\pi/2)$ are *undefined*.

Similarly, the cotangent and cosecant are not defined when $y = 0$. For instance, because $t = 0$ corresponds to $(x, y) = (1, 0)$, $\cot 0$ and $\csc 0$ are *undefined*.

The Trigonometric Functions

In Figure 1.17, the unit circle is divided into eight equal arcs, corresponding to *t*-values of

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \text{ and } 2\pi.$$

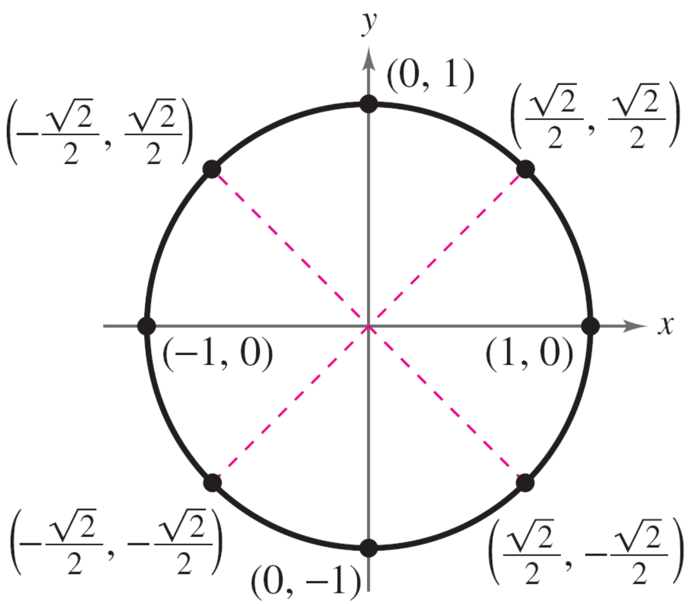


Figure 1.17

The Trigonometric Functions

Similarly, in Figure 1.18, the unit circle is divided into 12 equal arcs, corresponding to *t*-values of

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi.$$

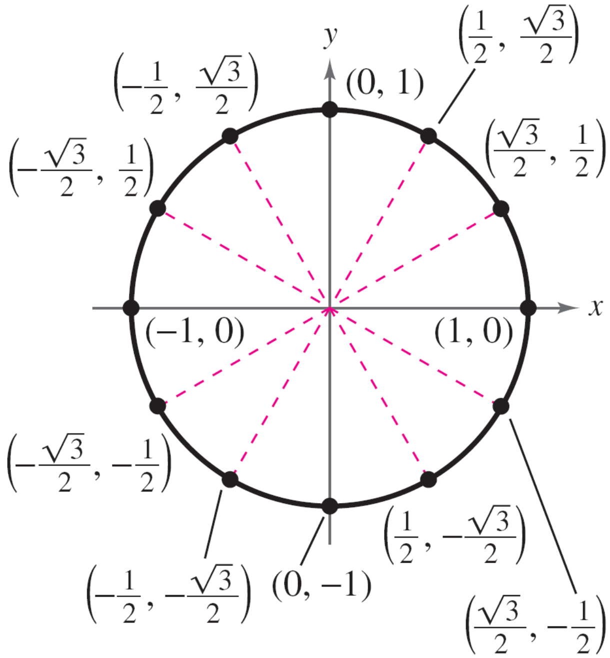


Figure 1.18

The Trigonometric Functions

To verify the points on the unit circle in Figure 1.17, note that $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ also lies on the line $y = x$. So, substituting x for y in the equation of the unit circle produces the following.

$$x^2 + x^2 = 1 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{\sqrt{2}}{2}$$

Because the point is in the first quadrant, and $y = x$, you have

$$x = \frac{\sqrt{2}}{2} \quad \text{and} \quad y = \frac{\sqrt{2}}{2}.$$

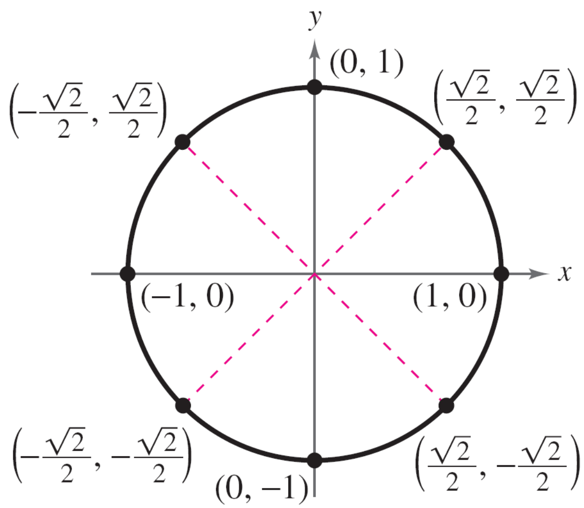


Figure 1.17

The Trigonometric Functions

You can use similar reasoning to verify the rest of the points in Figure 1.17 and the points in Figure 1.18.

Using the (x, y) coordinates in Figures 1.17 and 1.18, you can evaluate the trigonometric functions for common t-values.

Examples 1 demonstrates this procedure.

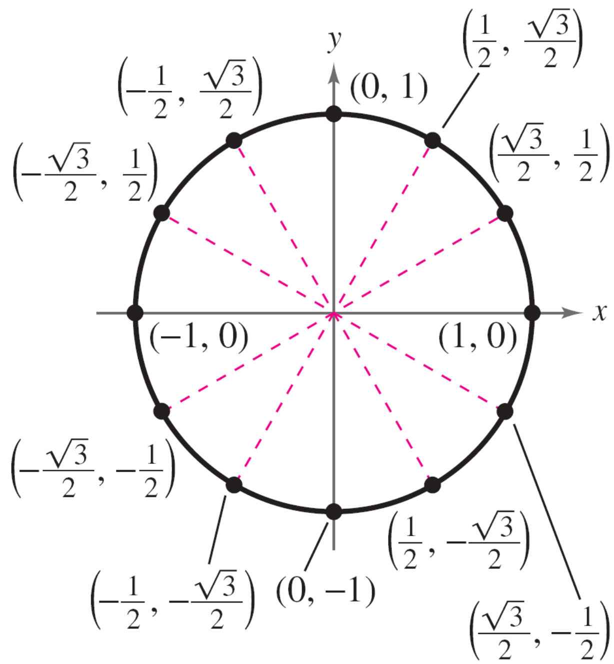


Figure 1.18

Example 1 – *Evaluating Trigonometric Functions*

Evaluate the six trigonometric functions at each real number.

a. $t = \frac{\pi}{6}$ **b.** $t = \frac{5\pi}{4}$ **c.** $t = \pi$ **d.** $t = -\frac{\pi}{3}$

Solution:

For each *t*-value, begin by finding the corresponding point (*x*, *y*) on the unit circle. Then use the definitions of trigonometric functions.

a. $t = \frac{\pi}{6}$ corresponds to the point $(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Example 1 – *Solution*

cont'd

$$\sin\frac{\pi}{6} = y = \frac{1}{2}$$

$$\cos\frac{\pi}{6} = x = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc\frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

Example 1 – *Solution*

cont'd

$$\sec \frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot \frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Example 1 – *Solution*

cont'd

b. $t = \dfrac{5\pi}{4}$ corresponds to the point (x, y) = $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$.

$$\sin\frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2}$$

$$\cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2}$$

$$\tan\frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

$$\csc\frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

Example 1 – *Solution*

cont'd

$$\sec \frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cot \frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

Example 1 – *Solution*

cont'd

c. $t = \pi$ corresponds to the point $(x, y) = (-1, 0)$.

$$\sin \pi = y = 0$$

$$\cos \pi = x = -1$$

$$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$$

Example 1 – *Solution*

cont'd

$$\csc \pi = \frac{1}{y}$$ is undefined.

$$\sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$$

$$\cot \pi = \frac{x}{y}$$ is undefined.

Example 1 – *Solution*

cont'd

d. Moving *clockwise* around the unit circle, it follows that $t = -\frac{\pi}{3}$ corresponds to the point $(x, y) = \left(1/2, -\sqrt{3}/2\right)$.

$$\sin\left(-\frac{\pi}{3}\right) = y = -\frac{\sqrt{3}}{2}$$

$$\cos\left(-\frac{\pi}{3}\right) = x = \frac{1}{2}$$

$$\tan\left(-\frac{\pi}{3}\right) = \frac{y}{x} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

Example 1 – *Solution*

cont'd

$$\csc\left(-\frac{\pi}{3}\right) = \frac{1}{y} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\sec\left(-\frac{\pi}{3}\right) = \frac{1}{x} = \frac{1}{1/2} = 2$$

$$\cot\left(-\frac{\pi}{3}\right) = \frac{x}{y} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Domain and Period of Sine and Cosine

Domain and Period of Sine and Cosine

The *domain* of the sine and cosine functions is the set of all real numbers. To determine the *range* of these two functions, consider the unit circle shown in Figure 1.19.

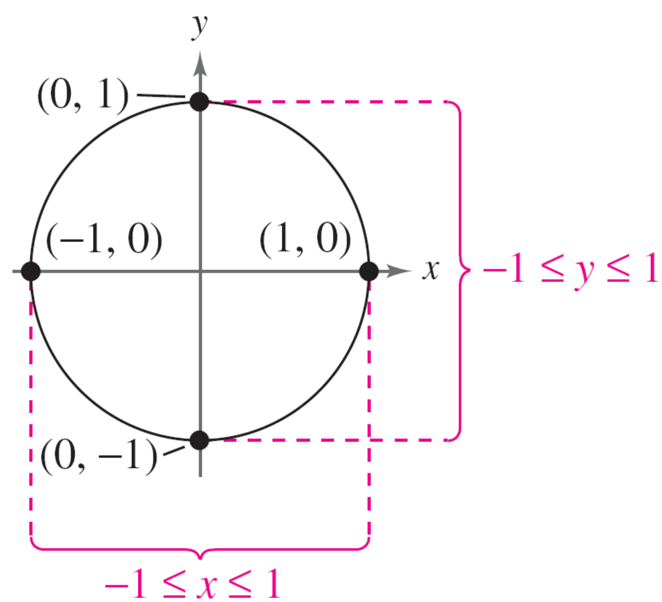


Figure 1.19

Domain and Period of Sine and Cosine

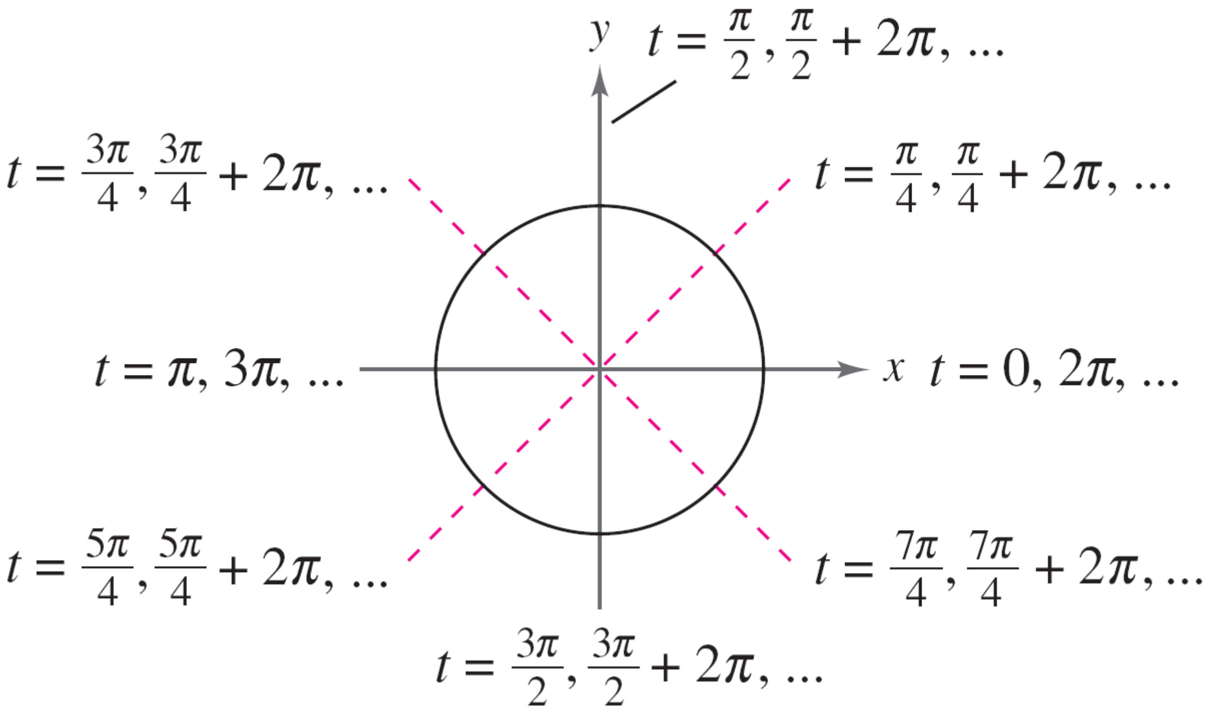
By definition, $\sin t = y$ and $\cos t = x$. Because (x, y) is on the unit circle, you know that $-1 \le y \le 1$ and $-1 \le x \le 1$. So, the values of sine and cosine also range between -1 and 1.

$$-1 \le y \le 1 \qquad \text{and} \qquad -1 \le x \le 1$$

$$-1 \le \sin t \le 1 \qquad\qquad -1 \le \cos t \le 1$$

Domain and Period of Sine and Cosine

Adding 2π to each value of t in the interval $[0, 2\pi]$ results in a revolution around the unit circle, as shown below.



Domain and Period of Sine and Cosine

The values of $\sin(t + 2\pi)$ and $\cos(t + 2\pi)$ correspond to those of $\sin t$ and $\cos t$.

Similar results can be obtained for repeated revolutions (positive or negative) on the unit circle. This leads to the general result

$$\sin(t + 2\pi n) = \sin t$$

and

$$\cos(t + 2\pi n) = \cos t$$

for any integer n and real number t. Functions that behave in such a repetitive (or cyclic) manner are called **periodic.**

Domain and Period of Sine and Cosine

Definition of Periodic Function

A function f is **periodic** when there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f. The smallest number c for which f is periodic is called the **period** of f.

We know that a function f is *even* when $f(-t) = f(t)$, and is *odd* when $f(-t) = -f(t)$.

Even and Odd Trigonometric Functions

The cosine and secant functions are *even*.

$$\cos(-t) = \cos t \qquad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$$\sin(-t) = -\sin t \qquad \csc(-t) = -\csc t$$

$$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$$

Example 2 – *Evaluating Sine and Cosine*

a. Because $\dfrac{13\pi}{6} = 2\pi + \dfrac{\pi}{6}$, you have

$$\begin{aligned}\sin\frac{13\pi}{6} &= \sin\left(2\pi + \frac{\pi}{6}\right)\\ &= \sin\frac{\pi}{6}\\ &= \frac{1}{2}.\end{aligned}$$

b. Because $-\dfrac{7\pi}{2} = -4\pi + \dfrac{\pi}{2}$, you have

$$\cos\left(-\frac{7\pi}{2}\right) = \cos\left(-4\pi + \frac{\pi}{2}\right)$$

Example 2 – *Evaluating Sine and Cosine* cont'd

$$= \cos \frac{\pi}{2}$$

$$= 0.$$

c. For $\sin t = \frac{4}{5}$, $\sin(-t) = -\frac{4}{5}$ because the sine function is odd.

Domain and Period of Sine and Cosine

When evaluating a trigonometric function with a calculator, you need to set the calculator to the desired *mode* of measurement (*degree* or *radian*).

Most calculators do not have keys for the cosecant, secant, and cotangent functions. To evaluate these functions, you can use the $\boxed{x^{-1}}$ key with their respective reciprocal functions: sine, cosine, and tangent.

Domain and Period of Sine and Cosine

For instance, to evaluate csc(π/8), use the fact that

$$\csc\frac{\pi}{8} = \frac{1}{\sin(\pi/8)}$$

and enter the following keystroke sequence in *radian* mode.

(SIN (π ÷ 8)) x^{-1} ENTER   Display 2.6131259

Example 3 – *Using a Calculator*

Function	**Mode**	**Calculator Keystrokes**	**Display**
a. $\sin \dfrac{2\pi}{3}$	Radian	SIN (2 π ÷ 3) ENTER	0.8660254
b. $\cot 1.5$	Radian	(TAN (1.5)) x^{-1} ENTER	0.0709148