1 Trigonometry

1.1 Radian and Degree Measure

Objectives

- Describe angles.
- Use radian measure.
- Use degree measure.
- Use angles to model and solve real-life problems.

Angles

Angles

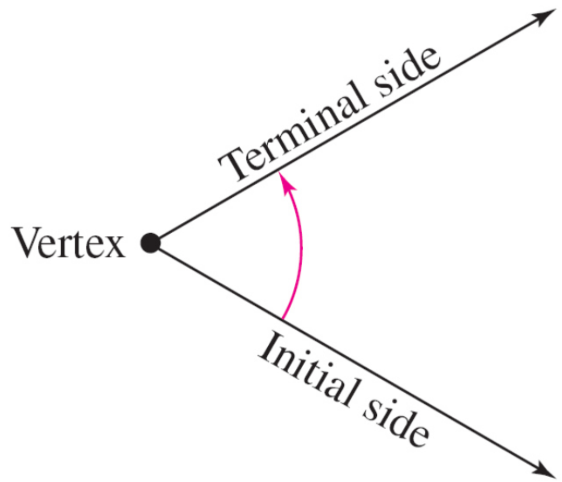
As derived from the Greek language, the word **trigonometry** means “measurement of triangles.”

Initially, trigonometry dealt with relationships among the sides and angles of triangles.

An **angle** is determined by rotating a ray (half-line) about its endpoint.

Angles

The starting position of the ray is the **initial side** of the angle, and the position after rotation is the **terminal side,** as shown in Figure 1.1.



Angle

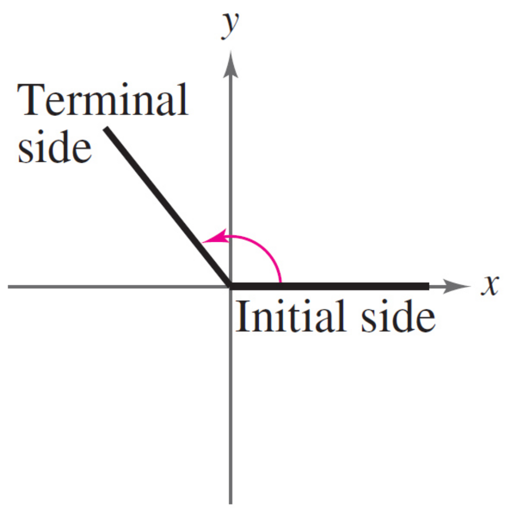
Figure 1.1

Angles

The endpoint of the ray is the **vertex** of the angle.

This perception of an angle fits a coordinate system in which the origin is the vertex and the initial side coincides with the positive *x*-axis.

Such an angle is in **standard position,** as shown in Figure 1.2.



Angle in standard position

Figure 1.2

Angles

Counterclockwise rotation generates **positive angles** and clockwise rotation generates **negative angles,** as shown in Figure 1.3.

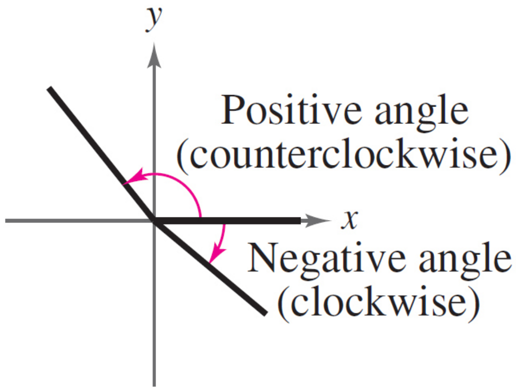
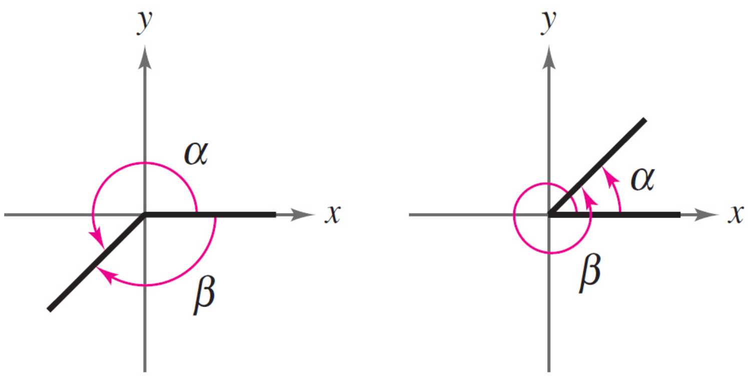


Figure 1.3

Angles are labeled with Greek letters such as α (alpha), β (beta), and θ (theta), as well as uppercase letters *A*, *B*, and *C*.

In Figure 1.4, note that angles α and β have the same initial and terminal sides. Such angles are **coterminal.**



Coterminal angles

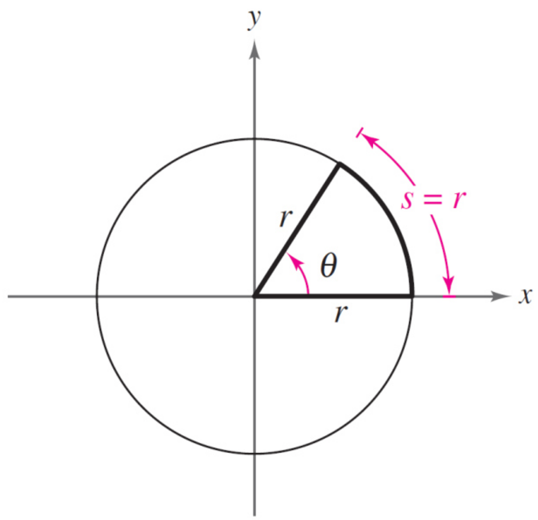
Figure 1.4

Radian Measure

Radian Measure

You determine the **measure of an angle** by the amount of rotation from the initial side to the terminal side. One way to measure angles is in *radians*. This type of measure is especially useful in calculus.

To define a radian, you can use a **central angle** of a circle, one whose vertex is the center of the circle, as shown in Figure 1.5.



Arc length = radius when θ = 1 radian

Figure 1.5

Radian Measure

Definition of Radian

One **radian** is the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle. See Figure 4.5. Algebraically, this means that

$$\theta = \frac{s}{r}$$

where θ is measured in radians. (Note that $\theta = 1$ when $s = r$.)

Because the circumference of a circle is $2\pi r$ units, it follows that a central angle of one full revolution (counterclockwise) corresponds to an arc length of

$$s = 2\pi r.$$

Radian Measure

Moreover, because $2\pi \approx 6.28$, there are just over six radius lengths in a full circle, as shown in Figure 1.6.

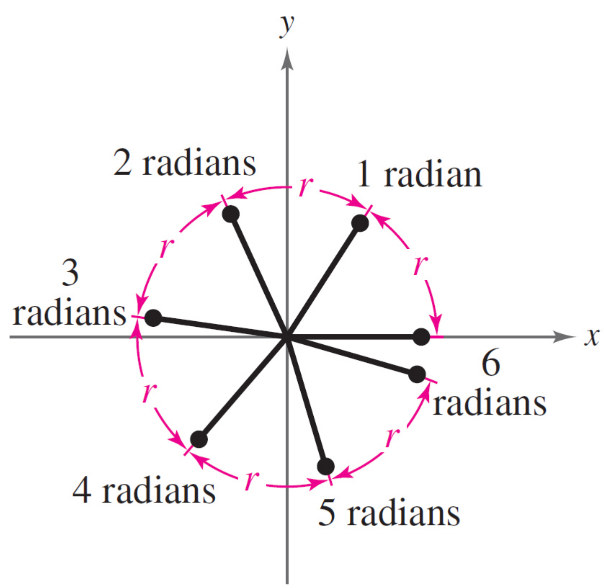


Figure 1.6

Because the units of measure for s and r are the same, the ratio s/r has no units—it is a real number.

Radian Measure

Because the measure of an angle of one full revolution is $s/r = 2\pi r/r = 2\pi$ radians, you can obtain the following.

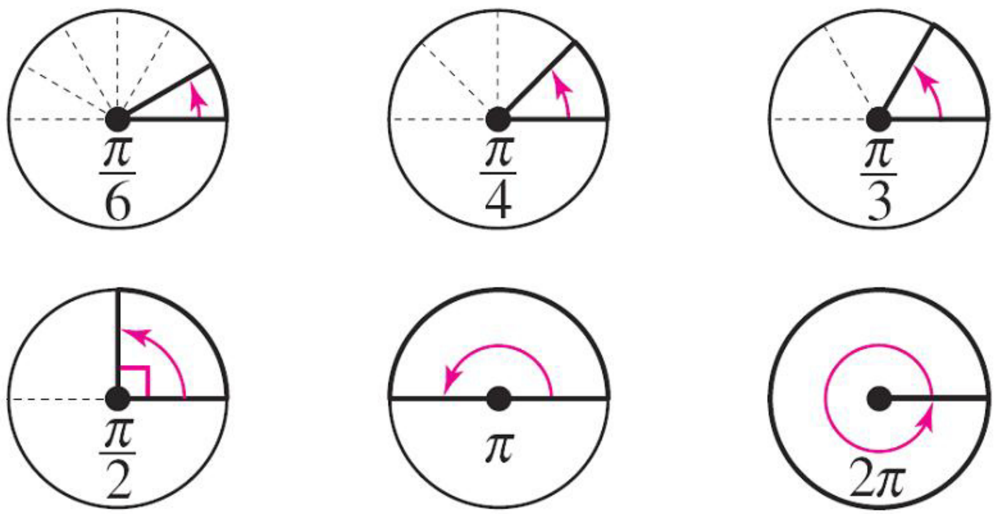
$$\frac{1}{2} \text{ revolution} = \frac{2\pi}{2} = \pi \text{ radians}$$

$$\frac{1}{4} \text{ revolution} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ radians}$$

$$\frac{1}{6} \text{ revolution} = \frac{2\pi}{6} = \frac{\pi}{3} \text{ radians}$$

Radian Measure

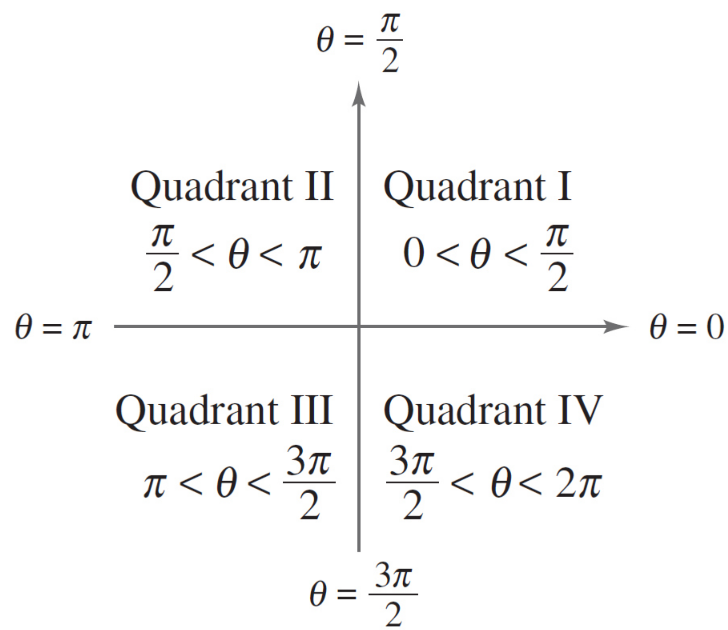
These and other common angles are shown below.



Radian Measure

We know that the four quadrants in a coordinate system are numbered I, II, III, and IV.

The figure below shows which angles between 0 and 2π lie in each of the four quadrants.



Note that angles between 0 and $\pi/2$ are **acute** angles and angles between $\pi/2$ and π are **obtuse** angles.

Radian Measure

Two angles are coterminal when they have the same initial and terminal sides. For instance, the angles 0 and 2π are coterminal, as are the angles $\pi/6$ and $13\pi/6$.

You can find an angle that is coterminal to a given angle θ by adding or subtracting 2π (one revolution), as demonstrated in Example 1.

A given angle θ has infinitely many coterminal angles. For instance, $\theta = \pi/6$ is coterminal with

$$\frac{\pi}{6} + 2n\pi$$

where n is an integer.

Example 1 – *Finding Coterminal Angles*

a. For the positive angle $13\pi/6$, subtract 2π to obtain a coterminal angle

$$\frac{13\pi}{6} - 2\pi = \frac{\pi}{6}.$$

See Figure 1.7.

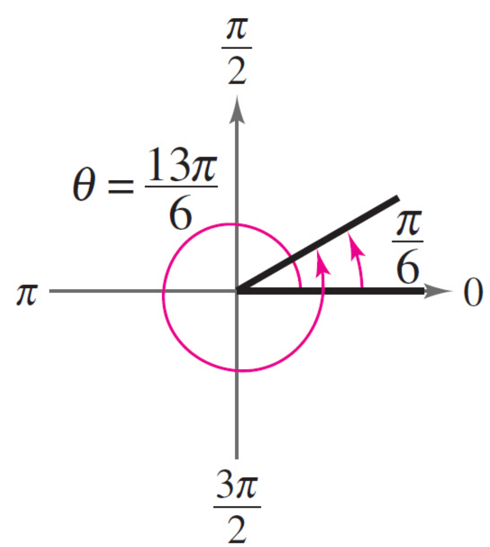


Figure 1.7

Example 1 – *Finding Coterminal Angles* cont'd

b. For the negative angle $-2\pi/3$, add 2π to obtain a coterminal angle

$$-\frac{2\pi}{3} + 2\pi = \frac{4\pi}{3}.$$

See Figure 1.8.

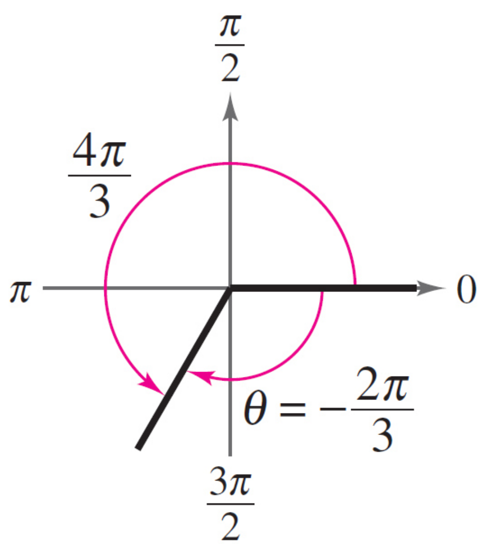


Figure 1.8

Radian Measure

Two positive angles α and β are **complementary** (complements of each other) when their sum is $\pi/2$.

Two positive angles are **supplementary** (supplements of each other) when their sum is π. See Figure 1.9.

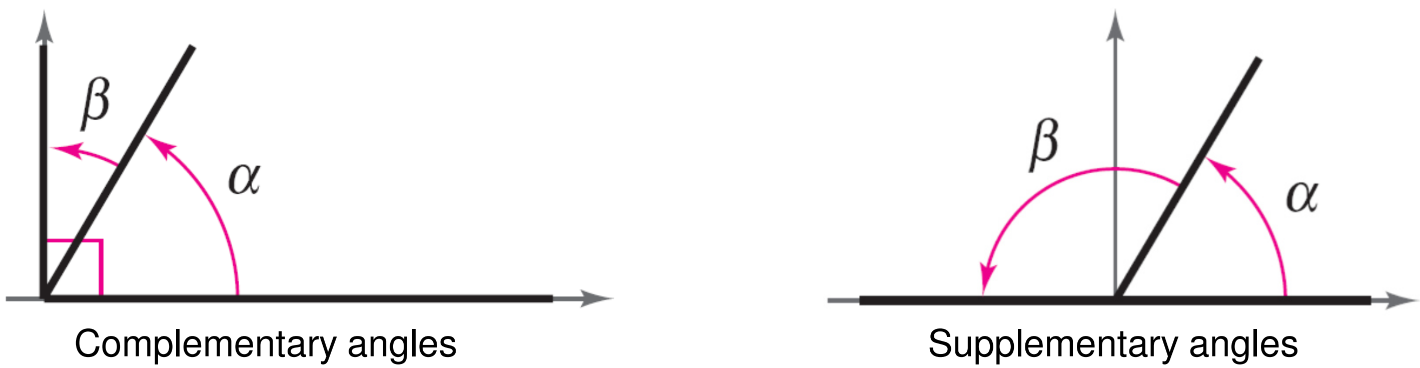


Figure 1.9

Degree Measure

Degree Measure

A second way to measure angles is in **degrees,** denoted by the symbol °.

A measure of one degree (1°) is equivalent to a rotation of $\frac{1}{360}$ of a complete revolution about the vertex.

To measure angles, it is convenient to mark degrees on the circumference of a circle, as shown in Figure 1.10.

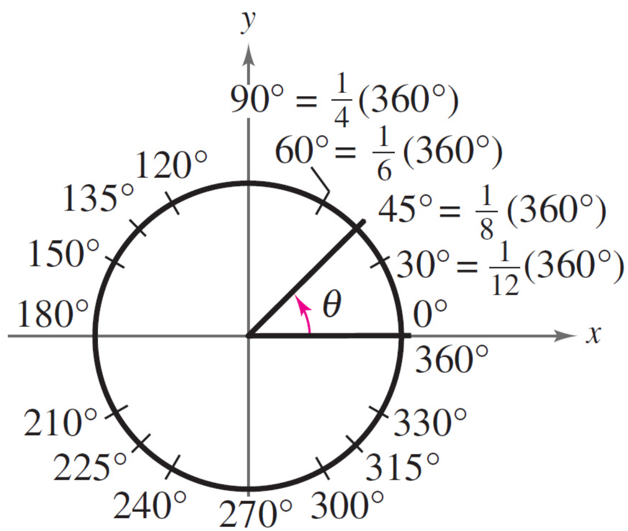


Figure 1.10

Degree Measure

So, a full revolution (counterclockwise) corresponds to 360°, a half revolution to 180°, a quarter revolution to 90°, and so on.

Because 2π radians corresponds to one complete revolution, degrees and radians are related by the equations

$$360° = 2\pi \text{ rad} \quad \text{and} \quad 180° = \pi \text{ rad}.$$

From the latter equation, you obtain

$$1° = \frac{\pi}{180} \text{ rad} \qquad \text{and} \qquad 1 \text{ rad} = \left(\frac{180°}{\pi}\right)$$

which lead to the conversion rules in the next slide.

Degree Measure

Conversions Between Degrees and Radians

1. To convert degrees to radians, multiply degrees by $\dfrac{\pi \text{ rad}}{180^\circ}$.

2. To convert radians to degrees, multiply radians by $\dfrac{180^\circ}{\pi \text{ rad}}$.

To apply these two conversion rules, use the basic relationship $\pi \text{ rad} = 180^\circ$. (See Figure 4.11.)

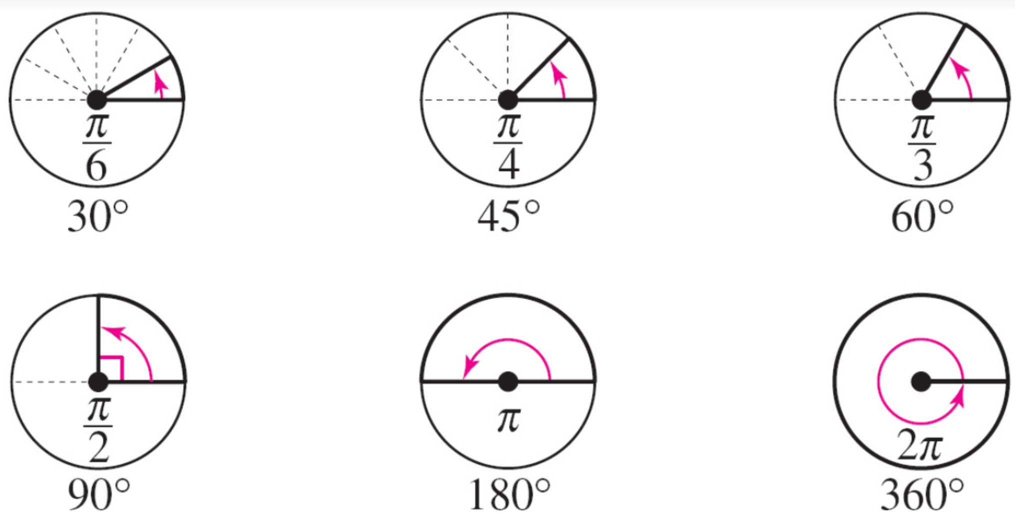


Figure 1.11

Degree Measure

When no units of angle measure are specified, *radian measure is implied*.

For instance, $\theta = 2$, implies that $\theta = 2$ radians.

Example 3 – *Converting from Degrees to Radians*

a. $135° = (135 \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right)$ Multiply by π rad / 180°.

$= \dfrac{3\pi}{4}$ radians

b. $540° = (540 \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right)$ Multiply by π rad / 180°.

$= 3\pi$ radians

Applications

Applications

The *radian measure* formula, $\theta = s/r$, can be used to measure arc length along a circle.

Arc Length

For a circle of radius r, a central angle θ intercepts an arc of length s given by

$$s = r\theta \qquad \text{Length of circular arc}$$

where θ is measured in radians. Note that if $r = 1$, then $s = \theta$, and the radian measure of θ equals the arc length.

Example 5 – *Finding Arc Length*

A circle has a radius of 4 inches. Find the length of the arc intercepted by a central angle of 240°, as shown in Figure 1.12.

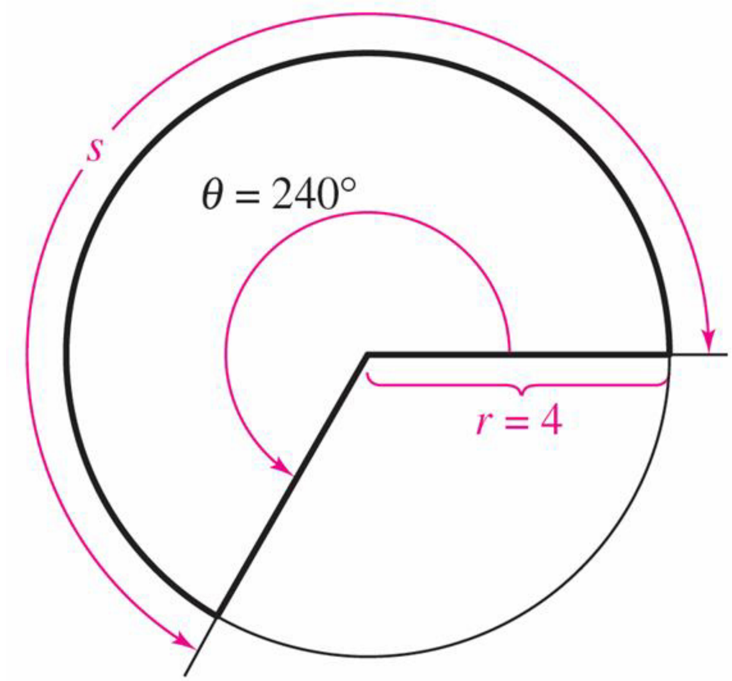


Figure 1.12

Example 5 – *Solution*

To use the formula $s = r\theta$, first convert 240° to radian measure.

$$\begin{aligned} 240^\circ &= (240\ \cancel{\text{deg}})\left(\frac{\pi\ \text{rad}}{180\ \cancel{\text{deg}}}\right) \\ &= \frac{4\pi}{3}\ \text{radians} \end{aligned}$$

Example 5 – *Solution*

cont'd

Then, using a radius of $r = 4$ inches, you can find the arc length to be

$$s = r\theta$$

$$= 4\left(\frac{4\pi}{3}\right)$$

$$\approx 16.76 \text{ inches.}$$

Note that the units for r determine the units for $r\theta$ because θ is given in radian measure, which has no units.

Applications

The formula for the length of a circular arc can help you analyze the motion of a particle moving at a *constant speed* along a circular path.

Linear and Angular Speeds

Consider a particle moving at a constant speed along a circular arc of radius r. If s is the length of the arc traveled in time t, then the **linear speed** v of the particle is

$$\text{Linear speed } v = \frac{\text{arc length}}{\text{time}} = \frac{s}{t}.$$

Moreover, if θ is the angle (in radian measure) corresponding to the arc length s, then the **angular speed** ω (the lowercase Greek letter omega) of the particle is

$$\text{Angular speed } \omega = \frac{\text{central angle}}{\text{time}} = \frac{\theta}{t}.$$

Applications

A **sector** of a circle is the region bounded by two radii of the circle and their intercepted arc (see Figure 1.15).

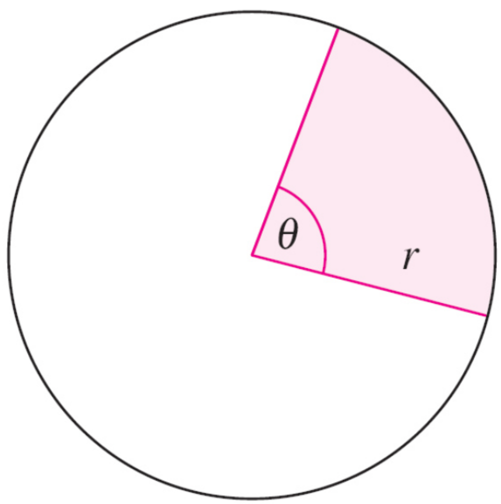


Figure 1.15

Applications

Area of a Sector of a Circle

For a circle of radius r, the area A of a sector of the circle with central angle θ is

$$A = \frac{1}{2}r^2\theta$$

where θ is measured in radians.