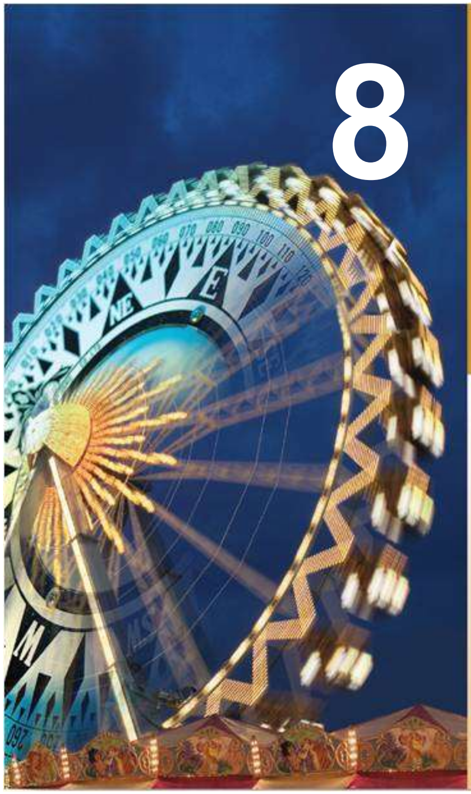

8 Complex Numbers and Polar Coordinates

SECTION 8.5

Polar Coordinates

Learning Objectives

1. Graph an ordered pair in polar coordinates.
2. Convert an ordered pair from polar to rectangular coordinates or vice-versa.
3. Express a polar equation in rectangular coordinates.
4. Express a rectangular equation in polar coordinates.

Example 1

A point lies at (4, 4) on a rectangular coordinate system. Give its address in polar coordinates (r, θ).

Solution:

We want to reach the same point by traveling r units on the terminal side of a standard position angle θ.

Figure 1 shows the point (4, 4), along with the distance r and angle θ.

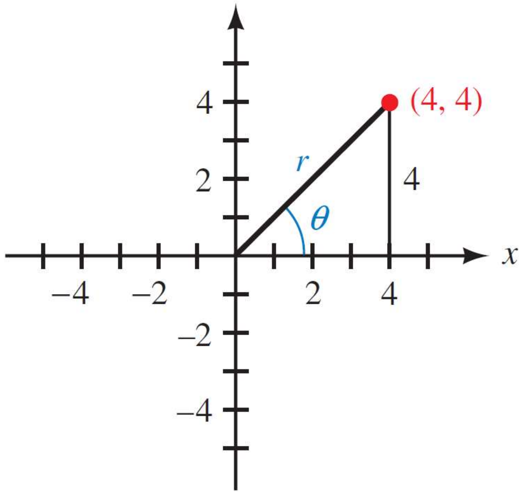

Figure 1

Example 1 – *Solution*

cont'd

The triangle formed is a 45°–45°–90° right triangle. Therefore, r is $4\sqrt{2}$, and θ is 45°.

In rectangular coordinates, the address of our point is (4, 4).

In polar coordinates, the address is ($4\sqrt{2}$, 45°) or ($4\sqrt{2}$, $\pi/4$) using radians.

Polar Coordinates

Now let's formalize our ideas about polar coordinates. The foundation of the polar coordinate system is a ray called the *polar axis*, whose initial point is called the *pole* (Figure 2).

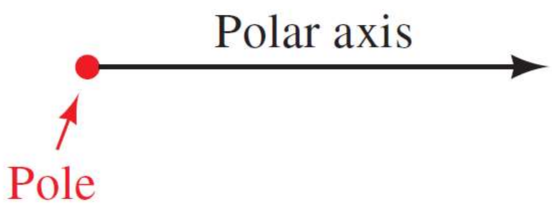

Figure 2

In the polar coordinate system, points are named by ordered pairs (r, θ) in which r is the directed distance from the pole on the terminal side of an angle θ, the initial side of which is the polar axis.

If the terminal side of θ has been rotated in a counterclockwise direction from the polar axis, then θ is a positive angle. Otherwise, θ is a negative angle.

Example 2

Graph the points (3, 45°), (2, –4π/3), (–4, π/3), and (–5, –210°) on a polar coordinate system.

Solution:

To graph (3, 45°), we locate the point that is 3 units from the origin along the terminal side of 45°, as shown in Figure 4.

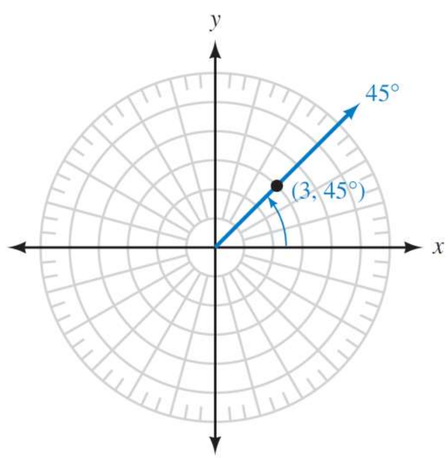

Figure 4

Example 2 – *Solution*

cont'd

The point $(2, -4\pi/3)$ is 2 units along the terminal side of $-4\pi/3$, as Figure 5 indicates.

As you can see from Figures 4 and 5, if r is positive, we locate the point (r, θ) along the terminal side of θ.

The next two points we will graph have negative values of r.

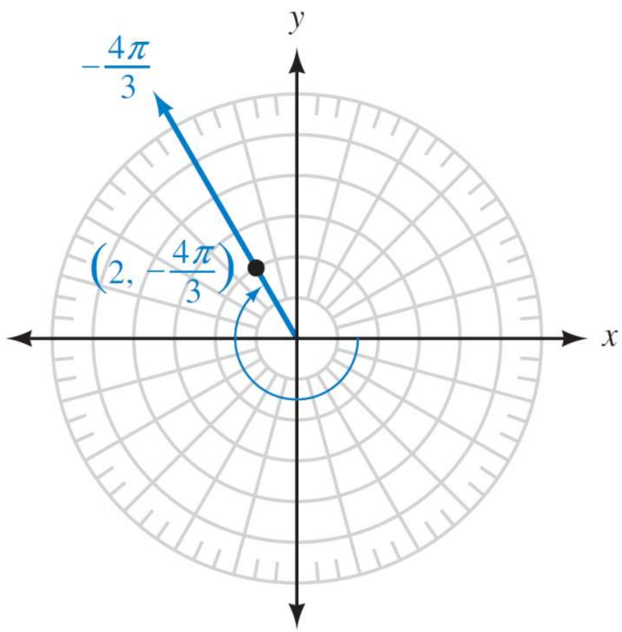

Figure 5

Example 2 – *Solution*

cont'd

To graph a point (r, θ) in which r is negative, we look for the point that is r units in the *opposite* direction indicated by the terminal side of θ.

That is, we extend a ray from the pole that is directly opposite the terminal side of θ and locate the point r units along this ray.

Example 2 – *Solution*

cont'd

To graph (–4, $\pi/3$), we first extend the terminal side of θ through the origin to create a ray in the opposite direction.

Then we locate the point that is 4 units from the origin along this ray, as shown in Figure 6.

The terminal side of θ has been drawn in blue, and the ray pointing in the opposite direction is shown in red.

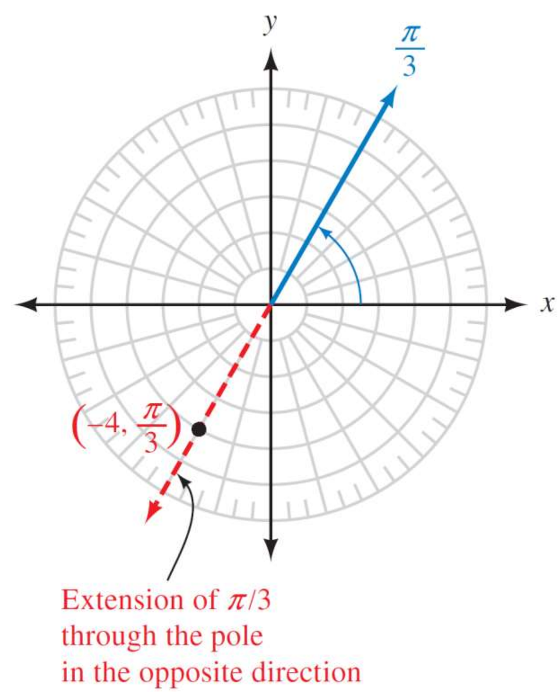

Figure 6

Example 2 – *Solution*

cont'd

To graph (–5, –210°), we look for the point that is 5 units from the origin along the ray in the opposite direction of –210° (Figure 7).

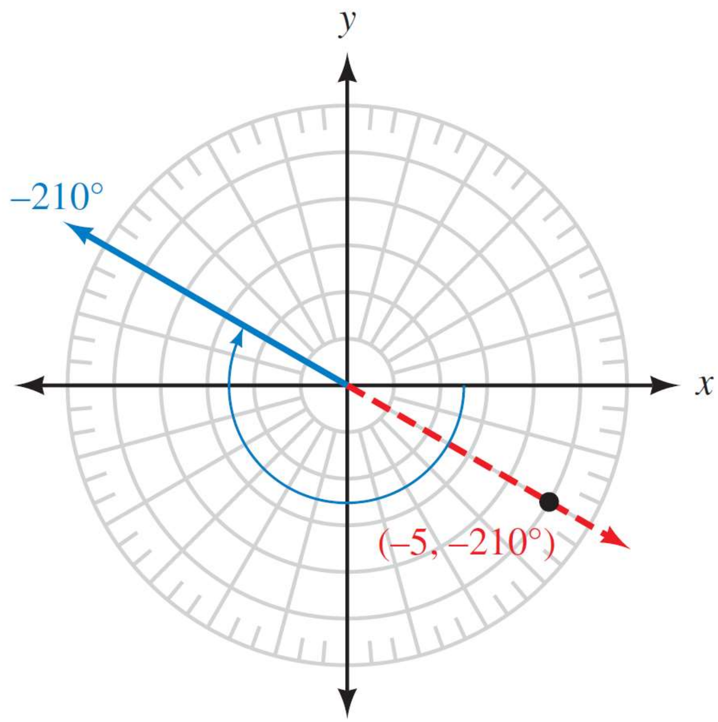

Figure 7

Polar Coordinates and Rectangular Coordinates

Polar Coordinates and Rectangular Coordinates

To derive the relationship between polar coordinates and rectangular coordinates, we consider a point *P* with rectangular coordinates (x, y) and polar coordinates (r, θ).

To convert back and forth between polar and rectangular coordinates, we simply use the relationships that exist among x, y, r, and θ in Figure 9.

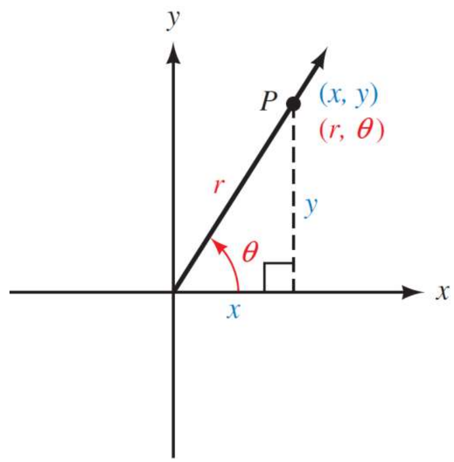

Figure 9

Polar Coordinates and Rectangular Coordinates

TO CONVERT RECTANGULAR COORDINATES TO POLAR COORDINATES

Let

$$r = \pm\sqrt{x^2 + y^2} \qquad \text{and} \qquad \tan\theta = \frac{y}{x}$$

where the sign of r and the choice of θ place the point (r, θ) in the same quadrant as (x, y).

TO CONVERT POLAR COORDINATES TO RECTANGULAR COORDINATES

Let

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

Polar Coordinates and Rectangular Coordinates

The process of converting to rectangular coordinates is simply a matter of substituting r and θ into the preceding equations.

To convert to polar coordinates, we have to choose θ and the sign of r so the point (r, θ) is in the same quadrant as the point (x, y).

Example 4

Convert to rectangular coordinates.

a. (4, 30°) **b.** $\left(-\sqrt{2}, \frac{3\pi}{4}\right)$ **c.** (3, 270°)

Solution:

To convert from polar coordinates to rectangular coordinates, we substitute the given values of *r* and θ into the equations

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

Here are the conversions for each point along with the graphs in both rectangular and polar coordinates.

Example 4(a) – *Solution*

cont'd

$$x = 4\cos 30^\circ \qquad\qquad y = 4\sin 30^\circ$$

$$= 4\left(\frac{\sqrt{3}}{2}\right) \qquad\qquad = 4\left(\frac{1}{2}\right)$$

$$= 2\sqrt{3} \qquad\qquad = 2$$

The point $(2\sqrt{3}, 2)$ in rectangular coordinates is equivalent to $(4, 30°)$ in polar coordinates. Figure 10 illustrates.

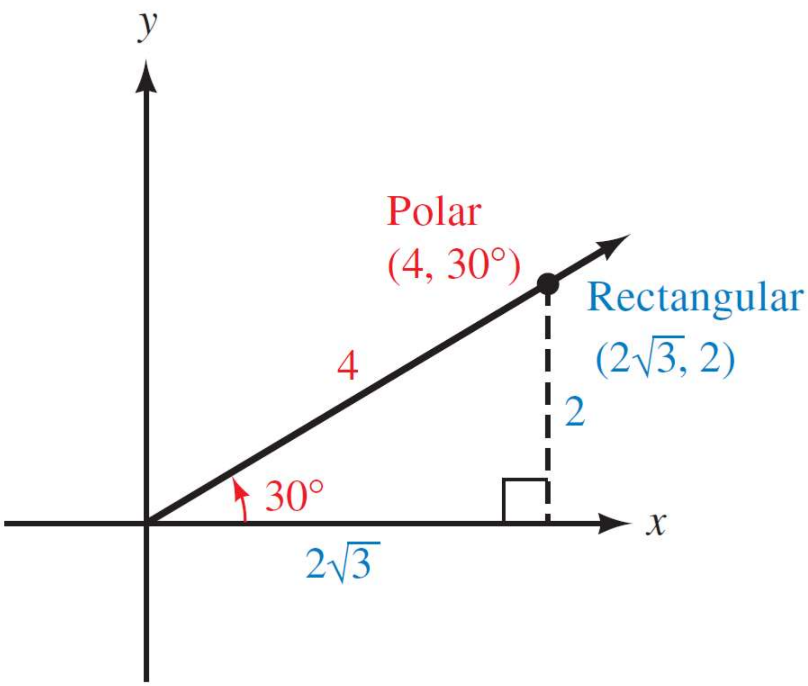

Figure 10

Example 4(b) – *Solution*

cont'd

$$x = -\sqrt{2}\cos\frac{3\pi}{4} \qquad y = -\sqrt{2}\sin\frac{3\pi}{4}$$

$$= -\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) \qquad = -\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)$$

$$= 1 \qquad = -1$$

The point (1, –1) in rectangular coordinates is equivalent to $(-\sqrt{2}, 3\pi/4)$ in polar coordinates. Figure 11 illustrates.

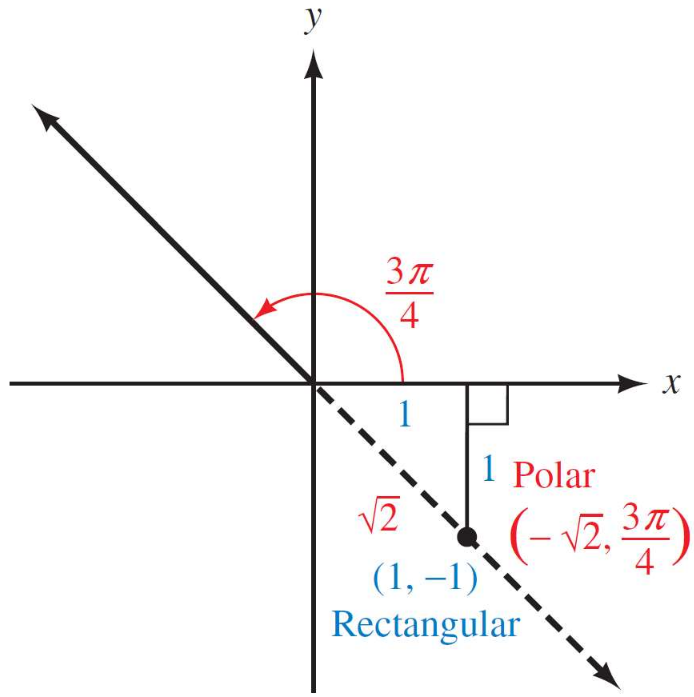

Figure 11

Example 4(c) – *Solution*

cont'd

$$x = 3\cos 270^\circ \qquad y = 3\sin 270^\circ$$

$$= 3(0) \qquad = 3(-1)$$

$$= 0 \qquad = -3$$

The point (0, –3) in rectangular coordinates is equivalent to (3, 270°) in polar coordinates. Figure 12 illustrates.

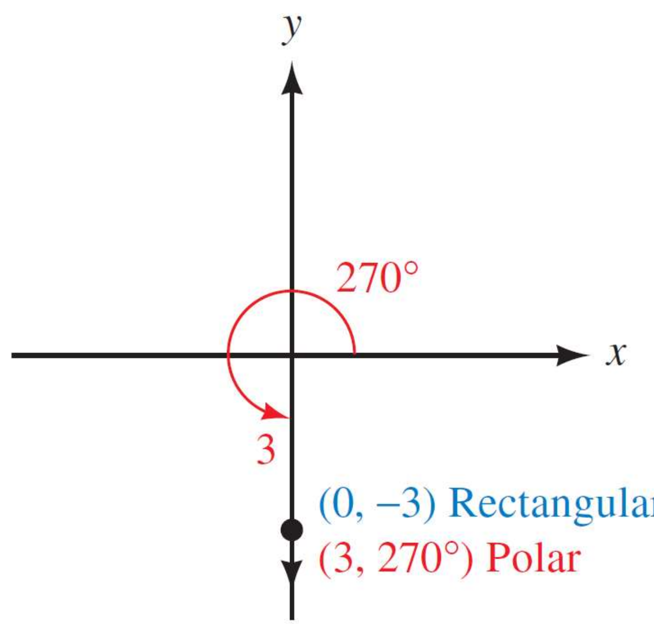

Figure 12

Example 5

Convert to polar coordinates.

a. (3, 3) **b.** (–2, 0) **c.** (–1, $\sqrt{3}$)

Solution:

a. Because *x* is 3 and *y* is 3, we have

$$r = \pm\sqrt{9+9}$$

$$= \pm 3\sqrt{2}$$

and $$\tan\theta = \frac{3}{3}$$

$$= 1$$

Example 5 – *Solution*

cont'd

Because (3, 3) is in QI, we can choose $r = 3\sqrt{2}$ and $\theta = 45°$ (or $\pi/4$), as shown in Figure 13.

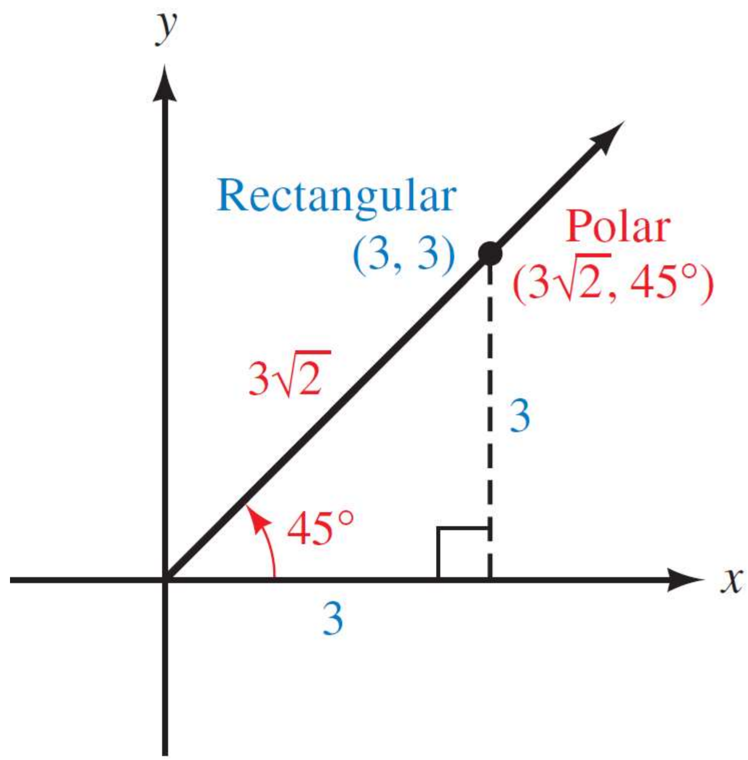

Figure 13

Example 5 – *Solution*

cont'd

Remember, there are an infinite number of ordered pairs in polar coordinates that name the point (3, 3).

The point $(3\sqrt{2}, 45°)$ is just one of them.

Generally, we choose *r* and θ so that *r* is positive and θ is between 0° and 360°.

b. We have *x* = –2 and *y* = 0, so

$$r = \pm\sqrt{4 + 0}$$

$$= \pm 2$$

Example 5 – *Solution*

cont'd

And

$$\tan \theta = \frac{0}{-2}$$

$$= 0$$

Because (–2, 0) is on the negative *x*-axis, we can choose $r = 2$ and $\theta = 180°$ (or π) to get the point (2, 180°). Figure 14 illustrates.

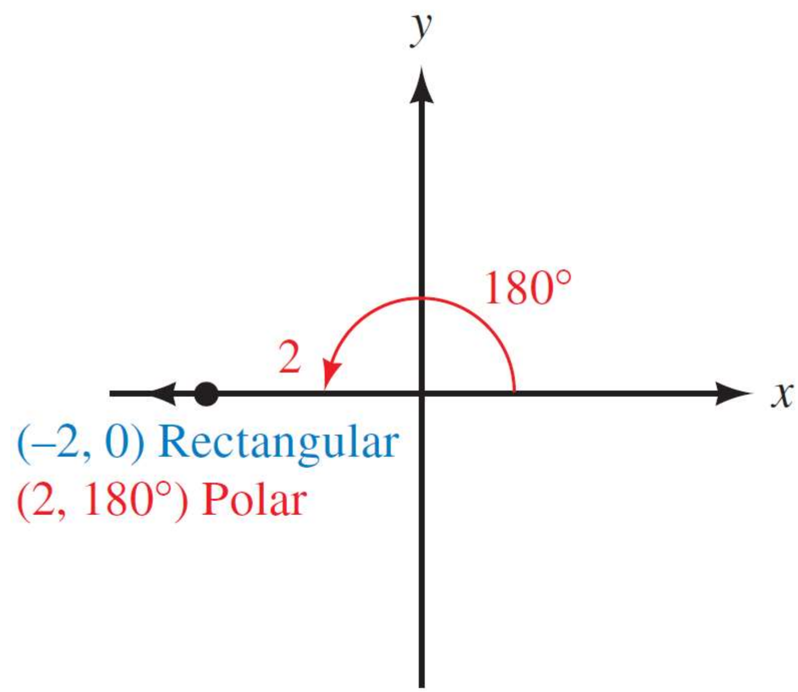

Figure 14

Example 5 – *Solution*

cont'd

c. Because $x = -1$ and $y = \sqrt{3}$, we have

$$r = \pm\sqrt{1 + 3}$$

$$= \pm 2$$

and

$$\tan\theta = \frac{\sqrt{3}}{-1}$$

Because $(-1, \sqrt{3})$ is in QII, we can let $r = 2$ and $\theta = 120°$ (or $2\pi/3$). In polar coordinates, the point is $(2, 120°)$. Figure 15 illustrates.

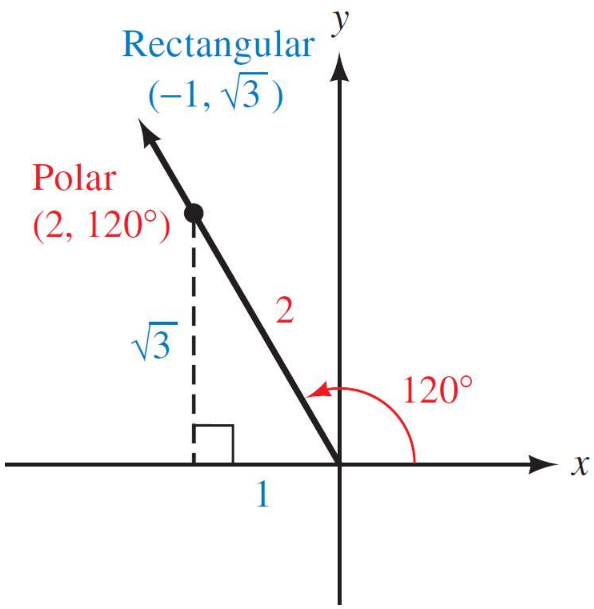

Figure 15

Equations in Polar Coordinates

Equations in Polar Coordinates

Equations in polar coordinates have variables r and θ instead of x and y.

Example 6

Change $r^2 = 9 \sin 2\theta$ to rectangular coordinates.

Solution:

Before we substitute to clear the equation of *r* and θ, we must use a double-angle identity to write sin 2θ in terms of sin θ and cos θ.

$$r^2 = 9 \sin 2\theta$$

$$r^2 = 9 \cdot 2 \sin \theta \cos \theta \qquad \text{Double-angle identity}$$

$$r^2 = 18 \cdot \frac{y}{r} \cdot \frac{x}{r} \qquad \text{Substitute } y/r \text{ for } \sin \theta \text{ and } x/r \text{ for } \cos \theta$$

Example 6 – *Solution*

cont'd

$$r^2 = \frac{18xy}{r^2}$$ Multiply

$$r^4 = 18xy$$ Multiply both sides by r^2

$$(x^2 + y^2)^2 = 18xy$$ Substitute $x^2 + y^2$ for r^2