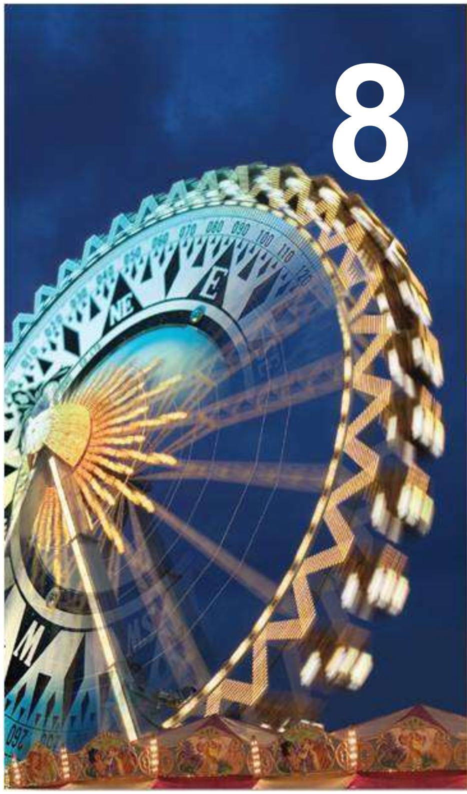

8 Complex Numbers and Polar Coordinates

SECTION 8.4

Roots of a Complex Number

Learning Objectives

1. Find both square roots of a complex number.
2. Find all *n*th roots of a complex number.
3. Graph the *n*th roots of a complex number.
4. Use roots to solve an equation.

Roots of a Complex Number

Every real (or complex) number has exactly two square roots, three cube roots, and four fourth roots.

In fact, every real (or complex) number has exactly *n* distinct *n*th roots, a surprising and attractive fact about numbers.

The key to finding these roots is trigonometric form for complex numbers.

Roots of a Complex Number

Suppose that *z* and *w* are complex numbers such that *w* is an *n*th root of *z*. That is,

$$w = \sqrt[n]{z}$$

THEOREM (ROOTS)

The nth roots of the complex number

$$z = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$$

are given by

$$\begin{aligned} w_k &= r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{360^\circ}{n}k\right) + i\sin\left(\frac{\theta}{n} + \frac{360^\circ}{n}k\right)\right] \\ &= r^{1/n}\operatorname{cis}\left(\frac{\theta}{n} + \frac{360^\circ}{n}k\right) \end{aligned}$$

where $k = 0, 1, 2, \ldots, n-1$. The root w_0, corresponding to $k = 0$, is called the *principal nth root* of z.

Example 1

Find the four fourth roots of $z = 16(\cos 60° + i \sin 60°)$.

Solution:

According to the formula given in our theorem on roots, the four fourth roots will be

$$w_k = 16^{1/4}\left[\cos\left(\frac{60°}{4} + \frac{360°}{4}k\right) + i \sin\left(\frac{60°}{4} + \frac{360°}{4}k\right)\right] \quad k = 0, 1, 2, 3$$

$$= 2[\cos(15° + 90°k) + i \sin(15° + 90°k)]$$

Example 1 – *Solution*

cont'd

Replacing k with 0, 1, 2, and 3, we have

$$w_0 = 2(\cos 15° + i \sin 15°) = 2 \text{ cis } 15° \qquad \text{when } k = 0$$

$$w_1 = 2(\cos 105° + i \sin 105°) = 2 \text{ cis } 105° \qquad \text{when } k = 1$$

$$w_2 = 2(\cos 195° + i \sin 195°) = 2 \text{ cis } 195° \qquad \text{when } k = 2$$

$$w_3 = 2(\cos 285° + i \sin 285°) = 2 \text{ cis } 285° \qquad \text{when } k = 3$$

Example 1 – *Solution*

cont'd

It is interesting to note the graphical relationship among these four roots, as illustrated in Figure 1.

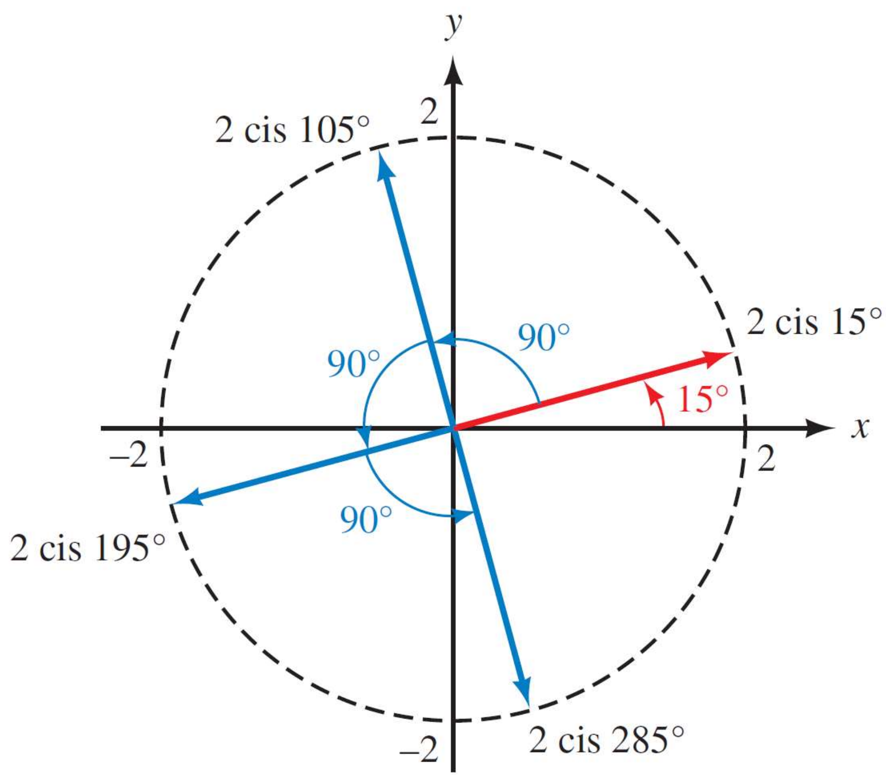

Figure 1

Example 1 – *Solution*

cont'd

Because the modulus for each root is 2, the terminal point of the vector used to represent each root lies on a circle of radius 2.

The vector representing the first root makes an angle of $\theta/n = 15°$ with the positive x-axis, then each following root has a vector that is rotated an additional $360°/n = 90°$ in the counterclockwise direction from the previous root.

Because the fraction $360°/n$ represents one complete revolution divided into n equal parts, the nth roots of a complex number will always be evenly distributed around a circle of radius $r^{1/n}$.

Example 2

Solve $x^3 + 1 = 0$.

Solution:

Adding –1 to each side of the equation, we have

$$x^3 = -1$$

The solutions to this equation are the cube roots of –1. We already know that one of the cube roots of –1 is –1. There are two other complex cube roots as well.

To find them, we write –1 in trigonometric form and then apply the formula from our theorem on roots.

Example 2 – *Solution*

cont'd

Writing –1 in trigonometric form, we have

$$-1 = 1(\cos \pi + i \sin \pi)$$

The 3 cube roots are given by

$$w_k = 1^{1/3}\left[\cos\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right) + i \sin\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)\right]$$

where k = 0, 1, and 2.

Replacing k with 0, 1, and 2 and then simplifying each result, we have

$$w_0 = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} \quad = \frac{1}{2} + \frac{\sqrt{3}}{2}i \qquad \text{when } k = 0$$

Example 2 – *Solution*

cont'd

$$w_1 = \cos \pi + i \sin \pi = -1 \qquad \text{when } k = 1$$

$$w_2 = \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2} i \qquad \text{when } k = 2$$

The graphs of these three roots, which are evenly spaced around the unit circle, are shown in Figure 3.

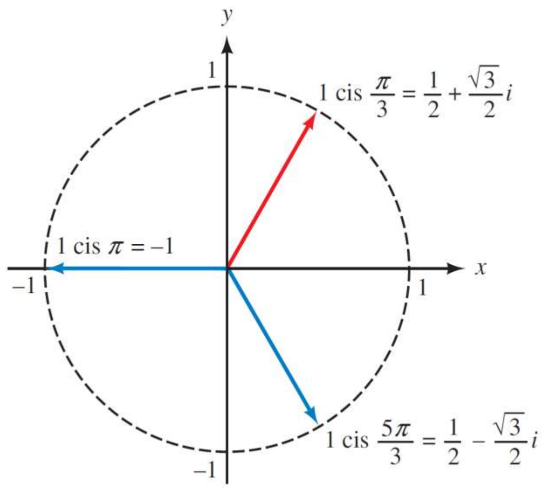

Figure 3

Example 2 – *Solution*

cont'd

Note that the two complex roots are conjugates. Let's check root w_0 by cubing it.

$$
\begin{aligned}
{w_0}^3 &= \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^3 \\
&= \cos 3 \cdot \frac{\pi}{3} + i \sin 3 \cdot \frac{\pi}{3} \\
&= \cos \pi + i \sin \pi \\
&= -1
\end{aligned}
$$

Example 3

Solve the equation $x^4 - 2\sqrt{3}x^2 + 4 = 0$.

Solution:

The equation is quadratic in x^2. We can solve for x^2 by applying the quadratic formula.

$$x^2 = \frac{2\sqrt{3} \pm \sqrt{12 - 4(1)(4)}}{2}$$

$$= \frac{2\sqrt{3} \pm \sqrt{-4}}{2}$$

$$= \frac{2\sqrt{3} \pm 2i}{2}$$

Example 3 – *Solution*

cont'd

$$= \sqrt{3} \pm i$$

The two solutions for x^2 are $\sqrt{3} + i$ and $\sqrt{3} - i$, which we write in trigonometric form as follows:

$$x^2 = \sqrt{3} + i \quad \text{or} \quad x^2 = \sqrt{3} - i$$

$$= 2(\cos 30^\circ + i \sin 30^\circ) \qquad = 2(\cos 330^\circ + i \sin 330^\circ)$$

Now each of these expressions has two square roots, each of which is a solution to our original equation.

When $x^2 = 2(\cos 30^\circ + i \sin 30^\circ)$

Example 3 – *Solution*

cont'd

$$x = 2^{1/2}\left[\cos\left(\frac{30^\circ}{2} + \frac{360^\circ}{2}k\right) + i\sin\left(\frac{30^\circ}{2} + \frac{360^\circ}{2}k\right)\right] \quad \text{for } k = 0 \text{ and } 1$$

$$= \sqrt{2}\text{ cis }15^\circ \qquad \text{when } k = 0$$

$$= \sqrt{2}\text{ cis }195^\circ \qquad \text{when } k = 1$$

When $x^2 = 2(\cos 330^\circ + i\sin 330^\circ)$

$$x = 2^{1/2}\left[\cos\left(\frac{330^\circ}{2} + \frac{360^\circ}{2}k\right) + i\sin\left(\frac{330^\circ}{2} + \frac{360^\circ}{2}k\right)\right] \quad \text{for } k = 0 \text{ and } 1$$

Example 3 – *Solution*

cont'd

$$= \sqrt{2}\text{ cis }165° \qquad \text{when } k = 0$$

$$= \sqrt{2}\text{ cis }345° \qquad \text{when } k = 1$$

Using a calculator and rounding to the nearest hundredth, we can write decimal approximations to each of these four solutions.

SOLUTIONS

Trigonometric Form		Decimal Approximation
$\sqrt{2}\text{ cis }15°$	$=$	$1.37 + 0.37i$
$\sqrt{2}\text{ cis }165°$	$=$	$-1.37 + 0.37i$
$\sqrt{2}\text{ cis }195°$	$=$	$-1.37 - 0.37i$
$\sqrt{2}\text{ cis }345°$	$=$	$1.37 - 0.37i$