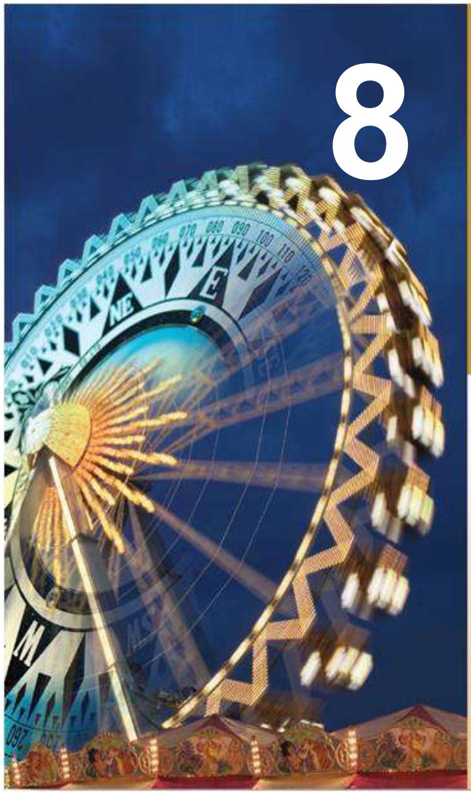

8 Complex Numbers and Polar Coordinates

SECTION 8.3

Products and Quotients in Trigonometric Form

Learning Objectives

1. Multiply complex numbers in trigonometric form.
2. Divide complex numbers in trigonometric form.
3. Use de Moivre’s Theorem to find powers of complex numbers.
4. Use trigonometric form to simplify expressions involving complex numbers.

Products and Quotients in Trigonometric Form

Multiplication and division with complex numbers becomes a very simple process when the numbers are written in trigonometric form.

Let’s state the rule for finding the product of two complex numbers written in trigonometric form as a theorem.

Products and Quotients in Trigonometric Form

THEOREM (MULTIPLICATION)

If

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 \operatorname{cis}\theta_1$$

and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 \operatorname{cis}\theta_2$$

are two complex numbers in trigonometric form, then their product, $z_1 z_2$, is

$$\begin{aligned} z_1 z_2 &= [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \\ &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \end{aligned}$$

In words: To multiply two complex numbers in trigonometric form, multiply the moduli and add the arguments.

Example 2

Find the product of $z_1 = 1 + i\sqrt{3}$ and $z_2 = -\sqrt{3} + i$ in standard form, and then write z_1 and z_2 in trigonometric form and find their product again.

Solution:

Leaving each complex number in standard form and multiplying we have

$$z_1 z_2 = (1 + i\sqrt{3})(-\sqrt{3} + i)$$

$$= -\sqrt{3} + i - 3i + i^2\sqrt{3}$$

$$= -2\sqrt{3} - 2i$$

Example 2 – *Solution*

cont'd

Changing z_1 and z_2 to trigonometric form and multiplying looks like this:

$$z_1 = 1 + i\sqrt{3} = 2(\cos 60° + i \sin 60°)$$

$$z_2 = -\sqrt{3} + i = 2(\cos 150° + i \sin 150°)$$

$$z_1 z_2 = [2(\cos 60° + i \sin 60°)][2(\cos 150° + i \sin 150°)]$$

$$= 4(\cos 210° + i \sin 210°)$$

Example 2 – *Solution*

cont'd

To compare our two products, we convert our product in trigonometric form to standard form.

$$4(\cos 210^\circ + i \sin 210^\circ) = 4\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= -2\sqrt{3} - 2i$$

As you can see, both methods of multiplying complex numbers produce the same result.

Products and Quotients in Trigonometric Form

The next theorem is an extension of the work we have done so far with multiplication.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$ is a complex number in trigonometric form and n is an integer, then

$$\begin{aligned} z^n &= [r(\cos\theta + i\sin\theta)]^n \\ &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n \operatorname{cis} n\theta \end{aligned}$$

Example 3

Find $(1 + i)^{10}$.

Solution:

First we write $1 + i$ in trigonometric form:

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Then we use de Moivre's Theorem to raise this expression to the 10th power.

$$(1 + i)^{10} = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{10}$$

Example 3 – *Solution*

cont'd

$$= (\sqrt{2})^{10}\left(\cos 10 \cdot \frac{\pi}{4} + i \sin 10 \cdot \frac{\pi}{4}\right)$$

$$= 32\left(\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2}\right)$$

which we can simplify to

$$= 32\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$$

because $\pi/2$ and $5\pi/2$ are coterminal.

Example 3 – *Solution*

cont'd

In standard form, our result is

$$= 32(0 + i)$$

$$= 32i$$

That is,

$$(1 + i)^{10} = 32i$$

Products and Quotients in Trigonometric Form

Multiplication with complex numbers in trigonometric form is accomplished by multiplying the moduli and adding the arguments, so we should expect that division is accomplished by dividing the moduli and subtracting the arguments.

Products and Quotients in Trigonometric Form

THEOREM (DIVISION)

If

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 \operatorname{cis}\theta_1$$

and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 \operatorname{cis}\theta_2$$

are two complex numbers in trigonometric form, then their quotient, z_1/z_2, is

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\ &= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \\ &= \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)\end{aligned}$$

Example 5

Divide $z_1 = 1 + i\sqrt{3}$ by $z_2 = \sqrt{3} + i$ and leave the answer in standard form. Then change each to trigonometric form and divide again.

Solution:

Dividing in standard form, we have

$$\frac{z_1}{z_2} = \frac{1 + i\sqrt{3}}{\sqrt{3} + i}$$

$$= \frac{1 + i\sqrt{3}}{\sqrt{3} + i} \cdot \frac{\boldsymbol{\sqrt{3} - i}}{\boldsymbol{\sqrt{3} - i}}$$

Example 5 – *Solution*

cont'd

$$= \frac{\sqrt{3} - i + 3i - i^2\sqrt{3}}{3 + 1}$$

$$= \frac{2\sqrt{3} + 2i}{4}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Changing z_1 and z_2 to trigonometric form,

$$z_1 = 1 + i\sqrt{3} = 2 \text{ cis } \frac{\pi}{3}$$

Example 5 – *Solution*

cont'd

$$z_2 = \sqrt{3} + i = 2 \text{ cis } \frac{\pi}{6}$$

and dividing again, we have

$$\frac{z_1}{z_2} = \frac{2 \text{ cis } (\pi/3)}{2 \text{ cis } (\pi/6)}$$

$$= \frac{2}{2} \text{ cis} \left(\frac{\pi}{3} - \frac{\pi}{6} \right)$$

$$= 1 \text{ cis } \frac{\pi}{6}$$

which, in standard form, is $\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$