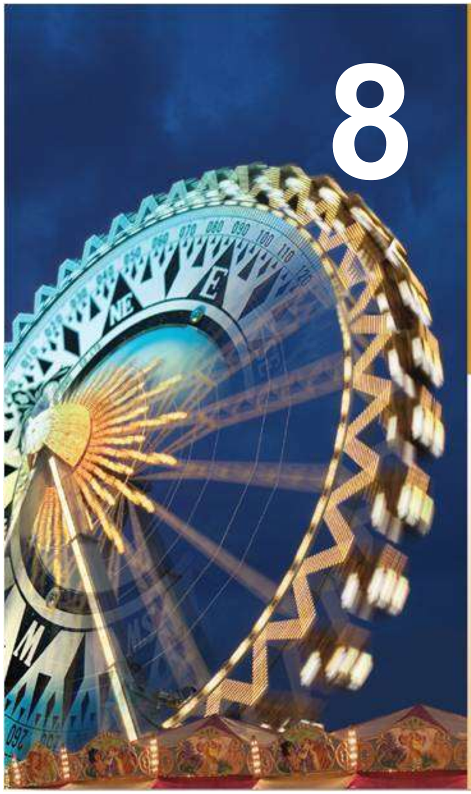

8 Complex Numbers and Polar Coordinates

SECTION 8.2

Trigonometric Form for Complex Numbers

Learning Objectives

1. Find the absolute value of a complex number.
2. Find the conjugate of a complex number.
3. Write a complex number in trigonometric form.
4. Convert a complex number from trigonometric form to standard form.

Trigonometric Form for Complex Numbers

In his book *Ars Magna*, Jerome Cardan gives a formula that can be used to solve certain cubic equations. Here it is in our notation:

If $x^3 = ax + b$

then $x = \sqrt[3]{\dfrac{b}{2} + \sqrt{\left(\dfrac{b}{2}\right)^2 - \left(\dfrac{a}{3}\right)^3}} + \sqrt[3]{\dfrac{b}{2} - \sqrt{\left(\dfrac{b}{2}\right)^2 - \left(\dfrac{a}{3}\right)^3}}$

This formula is known as Cardan's formula. In his book, Cardan attempts to use his formula to solve the equation

$$x^3 = 15x + 4$$

Trigonometric Form for Complex Numbers

This equation has the form $x^3 = ax + b$, where *a* = 15 and *b* = 4. Substituting these values for *a* and *b* in Cardan's formula, we have

$$x = \sqrt[3]{\frac{4}{2} + \sqrt{\left(\frac{4}{2}\right)^2 - \left(\frac{15}{3}\right)^3}} + \sqrt[3]{\frac{4}{2} - \sqrt{\left(\frac{4}{2}\right)^2 - \left(\frac{15}{3}\right)^3}}$$

$$= \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}}$$

$$= \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Cardan couldn't go any further than this because he didn't know what to do with $\sqrt{-121}$.

Trigonometric Form for Complex Numbers

Notice in his formula that if $(a/3)^3 > (b/2)^2$, then the result will be a negative number inside the square root.

In this section, we will take the first step in finding cube roots of complex numbers by learning how to write complex numbers in *trigonometric form*.

To graphically represent a complex number *a* + *bi*, we need a system which allows us to indicate the values of both *a* and *b.*

To do this, we set up a rectangular coordinate system much like the Cartesian coordinate system, except that we use the horizontal axis to indicate the real part and the vertical axis to indicate the imaginary part.

Trigonometric Form for Complex Numbers

We refer to the horizontal axis as the *real axis*, which represents the real part of a complex number, and the vertical axis as the *imaginary axis*, which represents the imaginary part. The resulting two-dimensional coordinate system is called the *complex plane*, or *Argand plane* (Figure 1).

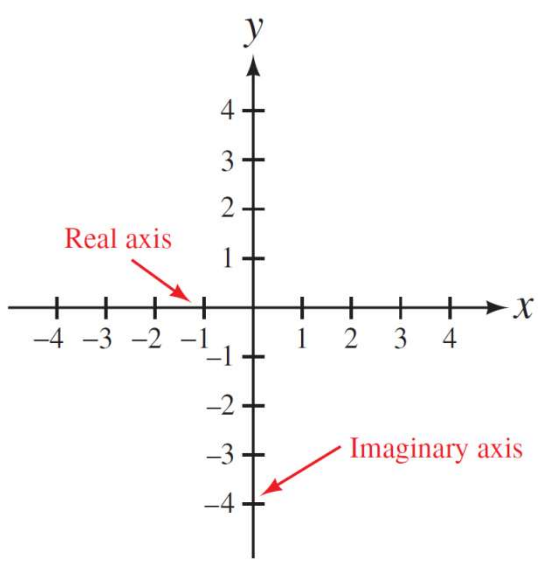

Figure 1

Trigonometric Form for Complex Numbers

DEFINITION

The graph of the complex number $x + yi$ is a vector (arrow) that extends from the origin out to the point (x, y) in the complex plane.

Example 1

Graph each complex number: $2 + 4i$, $-2 - 4i$, and $2 - 4i$.

Solution:

The graphs are shown in Figure 2. Notice how the graphs of $2 + 4i$ and $2 - 4i$, which are conjugates, have symmetry about the real axis, and that the graphs of $2 + 4i$ and $-2 - 4i$, which are opposites, have symmetry about the origin.

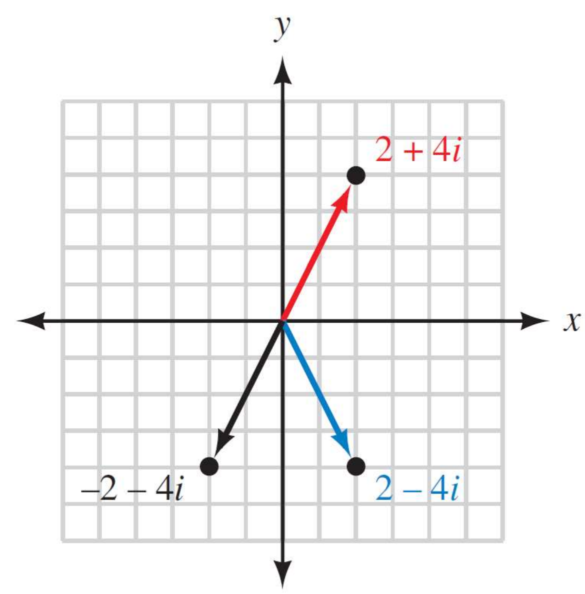

Figure 2

Trigonometric Form for Complex Numbers

DEFINITION

The *absolute value* or *modulus* of the complex number $z = x + yi$ is the distance from the origin to the point (x, y) in the complex plane. If this distance is denoted by r, then

$$r = |z| = |x + yi| = \sqrt{x^2 + y^2}$$

Example 3

Find the modulus of each of the complex numbers 5*i*, 7, and 3 + 4*i*.

Solution:

Writing each number in standard form and then applying the definition of modulus, we have

For $z = 5i = 0 + 5i, \quad r = |z| = |0 + 5i| = \sqrt{0^2 + 5^2} = 5$

For $z = 7 = 7 + 0i, \quad r = |z| = |7 + 0i| = \sqrt{7^2 + 0^2} = 7$

For $z = 3 + 4i, \quad r = |z| = |3 + 4i| = \sqrt{3^2 + 4^2} = 5$

Trigonometric Form for Complex Numbers

DEFINITION

The *argument* of the complex number $z = x + yi$, denoted $\arg(z)$, is the smallest positive angle θ from the positive real axis to the graph of z.

Figure 4 illustrates the relationships between the complex number $z = x + yi$, its graph, and the modulus r and argument θ of z.

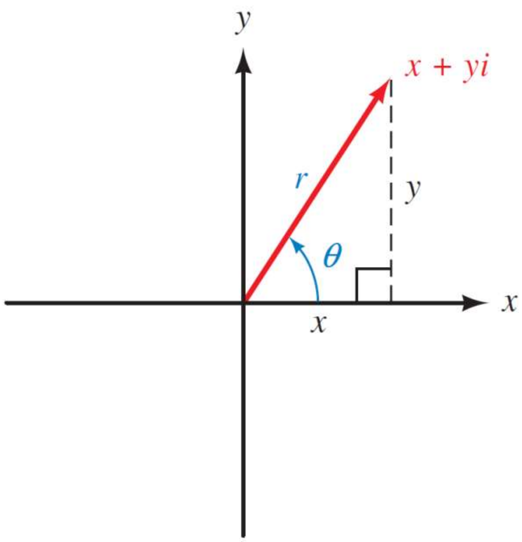

Figure 4

Trigonometric Form for Complex Numbers

From Figure 4 we see that

$$\cos\theta = \frac{x}{r} \qquad \text{or} \qquad x = r\cos\theta$$

and

$$\sin\theta = \frac{y}{r} \qquad \text{or} \qquad y = r\sin\theta$$

Trigonometric Form for Complex Numbers

We can use this information to write *z* in terms of *r* and θ.

$$
\begin{aligned}
z &= x + yi \\
&= r\cos\theta + (r\sin\theta)i \\
&= r\cos\theta + ri\sin\theta \\
&= r(\cos\theta + i\sin\theta)
\end{aligned}
$$

This last expression is called the *trigonometric form* for *z*, which can be abbreviated as *r* cis θ.

Trigonometric Form for Complex Numbers

The formal definition follows.

DEFINITION

If $z = x + yi$ is a complex number in standard form, then the *trigonometric form* for z is given by

$$z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$$

where r is the modulus of z and θ is the argument of z.

We can convert back and forth between standard form and trigonometric form by using the relationships that follow:

For $\quad z = x + yi = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$

$$r = \sqrt{x^2 + y^2} \qquad \text{and } \theta \text{ is such that}$$

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

Example 4

Write $z = -1 + i$ in trigonometric form.

Solution:

We have $x = -1$ and $y = 1$; therefore,

$$r = \sqrt{(-1)^2 + 1^2}$$

$$= \sqrt{2}$$

Example 4 – *Solution*

cont'd

Angle θ is the smallest positive angle for which

$$\cos\theta = \frac{x}{r} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

and

$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Therefore, θ must be 135°, or $3\pi/4$ radians.

Example 4 – *Solution*

cont'd

Using these values of r and θ in the formula for trigonometric form, we have

$$z = r(\cos\theta + i\sin\theta)$$

$$= \sqrt{2}(\cos 135^\circ + i\sin 135^\circ)$$

$$= \sqrt{2}\operatorname{cis} 135^\circ$$

In radians, $z = \sqrt{2}\operatorname{cis}(3\pi/4)$.

Example 4 – *Solution*

cont'd

The graph of z is shown in Figure 5.

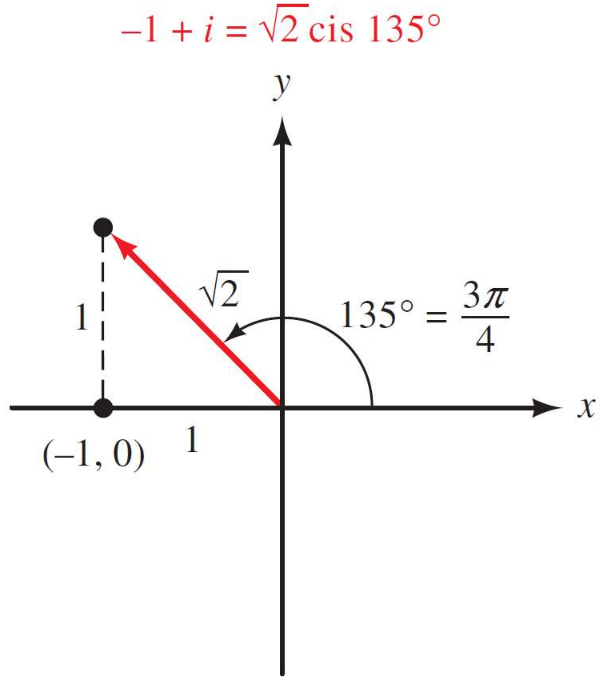

Figure 5

Example 5

Write z = 2 cis 60° in standard form.

Solution:

Using exact values for cos 60° and sin 60°, we have

$$z = 2 \text{ cis } 60^\circ$$

$$= 2(\cos 60^\circ + i \sin 60^\circ)$$

$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= 1 + i\sqrt{3}$$

Example 5 – *Solution*

cont'd

The graph of *z* is shown in Figure 6.

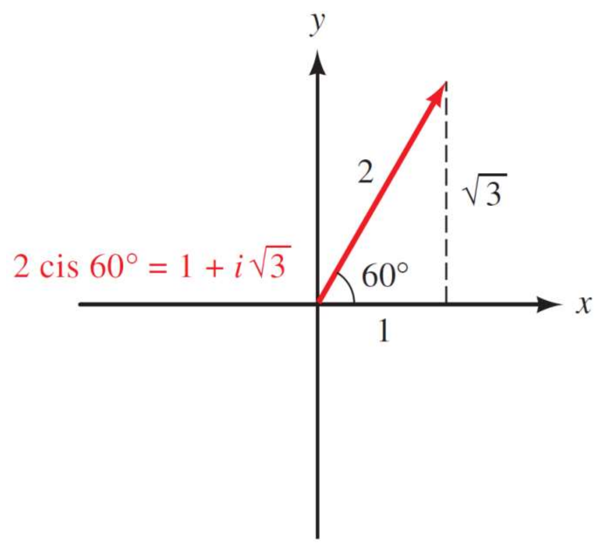

Figure 6