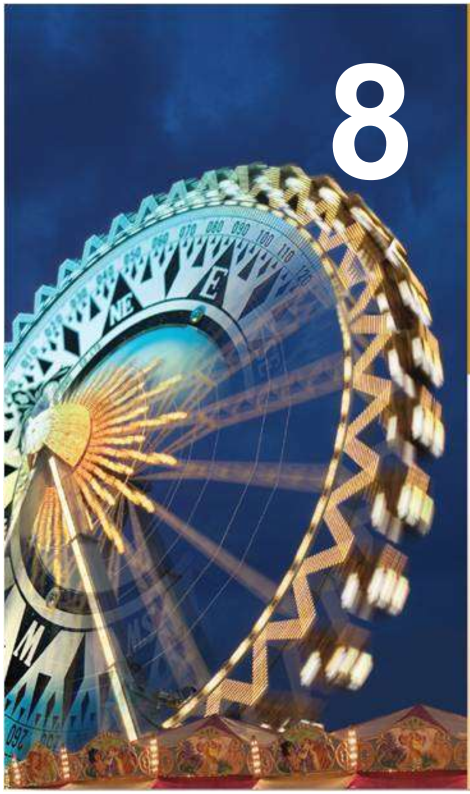

8 Complex Numbers and Polar Coordinates

SECTION 8.1

Complex Numbers

Learning Objectives

1. Express the square root of a negative number as an imaginary number.
2. Add and subtract complex numbers.
3. Simplify powers of *i*.
4. Multiply and divide complex numbers.

Complex Numbers

One of the problems posed by Cardan in *Ars Magna* is the following:

> If someone says to you, divide 10 into two parts, one of which multiplied into the other shall produce 40, it is evident that this case or question is impossible.

Using our present-day notation, this problem can be solved with a system of equations:

$$x + y = 10$$

$$xy = 40$$

Complex Numbers

Solving the first equation for *y*, we have $y = 10 - x$. Substituting this value of *y* into the second equation, we have

$$x(10 - x) = 40$$

The equation is quadratic. We write it in standard form and apply the quadratic formula.

$$0 = x^2 - 10x + 40$$

$$x = \frac{10 \pm \sqrt{100 - 4(1)(40)}}{2}$$

$$= \frac{10 \pm \sqrt{100 - 160}}{2}$$

Complex Numbers

$$= \frac{10 \pm \sqrt{-60}}{2}$$

$$= \frac{10 \pm 2\sqrt{-15}}{2}$$

$$= 5 \pm \sqrt{-15}$$

This is as far as Cardan could take the problem because he did not know what to do with the square root of a negative number.

We handle this situation by using *complex numbers*.

Complex Numbers

Our work with complex numbers is based on the following definition.

DEFINITION

The number i, called the *imaginary unit,* is such that $i^2 = -1$. (That is, i is the number whose square is -1.)

The number *i* is not a real number. We can use it to write square roots of negative numbers without a negative sign. To do so, we reason that if *a* > 0, then $\sqrt{-a} = \sqrt{ai^2} = i\sqrt{a}$.

Example 1

Write each expression in terms of *i*.

a. $\sqrt{-9}$ **b.** $\sqrt{-12}$ **c.** $\sqrt{-17}$

Solution:

a. $\sqrt{-9} = i\sqrt{9} = 3i$

b. $\sqrt{-12} = i\sqrt{12} = 2i\sqrt{3}$

c. $\sqrt{-17} = i\sqrt{17}$

Complex Numbers

Next, we use *i* to write a definition for complex numbers.

DEFINITION

A *complex number* is any number that can be written in the form

$$a + bi$$

where a and b are real numbers and $i^2 = -1$. The form $a + bi$ is called *standard form* for complex numbers. The number a is called the *real part* of the complex number. The number b is called the *imaginary part* of the complex number. If $b = 0$, then $a + bi = a$, which is a real number. If $a = 0$ and $b \neq 0$, then $a + bi = bi$, which is called an *imaginary number*.

Example 2

a. The number $3 + 2i$ is a complex number in standard form. The number 3 is the real part, and the number 2 (not $2i$) is the imaginary part.

b. The number $-7i$ is a complex number because it can be written as $0 + (-7)i$. The real part is 0. The imaginary part is -7. The number $-7i$ is also an imaginary number since $a = 0$ and $b \neq 0$.

c. The number 4 is a complex number because it can be written as $4 + 0i$. The real part is 4 and the imaginary part is 0.

Complex Numbers

From part *c* in Example 2, it is apparent that real numbers are also complex numbers.

The real numbers are a subset of the complex numbers.

Equality for Complex Numbers

Equality for Complex Numbers

DEFINITION

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. That is, for real numbers a, b, c, and d,

$$a + bi = c + di \qquad \text{if and only if} \qquad a = c \text{ and } b = d$$

Example 3

Find x and y if $(-3x - 9) + 4i = 6 + (3y - 2)i$.

Solution:

The real parts are $-3x - 9$ and 6. The imaginary parts are 4 and $3y - 2$.

$$-3x - 9 = 6 \qquad \text{and} \qquad 4 = 3y - 2$$

$$-3x = 15 \qquad\qquad 6 = 3y$$

$$x = -5 \qquad\qquad y = 2$$

Addition and Subtraction of Complex Numbers

Addition and Subtraction of Complex Numbers

DEFINITION

If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are complex numbers, then the sum and difference of z_1 and z_2 are defined as follows:

$$\begin{aligned} z_1 + z_2 &= (a_1 + b_1 i) + (a_2 + b_2 i) \\ &= (a_1 + a_2) + (b_1 + b_2)i \end{aligned}$$

$$\begin{aligned} z_1 - z_2 &= (a_1 + b_1 i) - (a_2 + b_2 i) \\ &= (a_1 - a_2) + (b_1 - b_2)i \end{aligned}$$

Example 4

If $z_1 = 3 - 5i$ and $z_2 = -6 - 2i$, find $z_1 + z_2$ and $z_1 - z_2$.

Solution:

$$z_1 + z_2 = (3 - 5i) + (-6 - 2i)$$

$$= -3 - 7i$$

$$z_1 - z_2 = (3 - 5i) - (-6 - 2i)$$

$$= 9 - 3i$$

Powers of *i*

Powers of *i*

If we assume the properties of exponents hold when the base is *i*, we can write any integer power of *i* as *i*, –1, –*i*, or 1. Using the fact that $i^2 = -1$, we have

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = i^2 \cdot i = -1(i) = -i$$

$$i^4 = i^2 \cdot i^2 = -1(-1) = 1$$

Powers of *i*

Because $i^4 = 1$, i^5 will simplify to i and we will begin repeating the sequence i, -1, $-i$, 1 as we increase our exponent by one each time.

$$i^5 = i^4 \cdot i = 1(i) = i$$

$$i^6 = i^4 \cdot i^2 = 1(-1) = -1$$

$$i^7 = i^4 \cdot i^3 = 1(-i) = -i$$

$$i^8 = i^4 \cdot i^4 = 1(1) = 1$$

We can simplify higher powers of i by writing them in terms of i^4 since i^4 is always 1.

Example 5

Simplify each power of *i*.

Solution:

a. $i^{20} = (i^4)^5 = 1^5 = 1$

b. $i^{23} = (i^4)^5 \cdot i^3 = 1(-i) = -i$

c. $i^{30} = (i^4)^7 \cdot i^2 = 1(-1) = -1$

Multiplication and Division with Complex Numbers

Multiplication and Division with Complex Numbers

DEFINITION

If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are complex numbers, then their product is defined as follows:

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i \end{aligned}$$

Example 7

Multiply $(4 - 5i)(4 + 5i)$.

Solution:

This product has the form $(a - b)(a + b)$, which we know results in the difference of two squares $a^2 - b^2$

$$
\begin{aligned}
(4 - 5i)(4 + 5i) &= 4^2 - (5i)^2 \\
&= 16 - 25i^2 \\
&= 16 - 25(-1) \\
&= 16 + 25 \\
&= 41
\end{aligned}
$$

Multiplication and Division with Complex Numbers

The product of the two complex numbers 4 – 5*i* and 4 + 5*i* is the real number 41. This fact is very useful and leads to the following definition.

DEFINITION

The complex numbers $a + bi$ and $a - bi$ are called *complex conjugates*. Their product is the real number $a^2 + b^2$. Here's why:

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - (bi)^2 \\ &= a^2 - b^2 i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2\end{aligned}$$

The fact that the product of two complex conjugates is a real number is the key to division with complex numbers.

Example 8

Divide $\dfrac{5i}{2 - 3i}$.

Solution:

We want to find a complex number in standard form that is equivalent to the quotient $5i/(2 - 3i)$. To do so, we need to replace the denominator with a real number.

We can accomplish this by multiplying both the numerator and the denominator by $2 + 3i$, which is the conjugate of $2 - 3i$.

$$\frac{5i}{2 - 3i} = \frac{5i}{2 - 3i} \cdot \frac{(\mathbf{2 + 3i})}{(\mathbf{2 + 3i})}$$

Example 8 – *Solution*

cont'd

$$= \frac{5i(2 + 3i)}{(2 - 3i)(2 + 3i)}$$

$$= \frac{10i + 15i^2}{4 - 9i^2}$$

$$= \frac{10i + 15(-1)}{4 - 9(-1)}$$

$$= \frac{-15 + 10i}{13}$$

$$= -\frac{15}{13} + \frac{10}{13}i$$

Notice that we have written our answer in standard form. The real part is –15/13 and the imaginary part is 10/13.