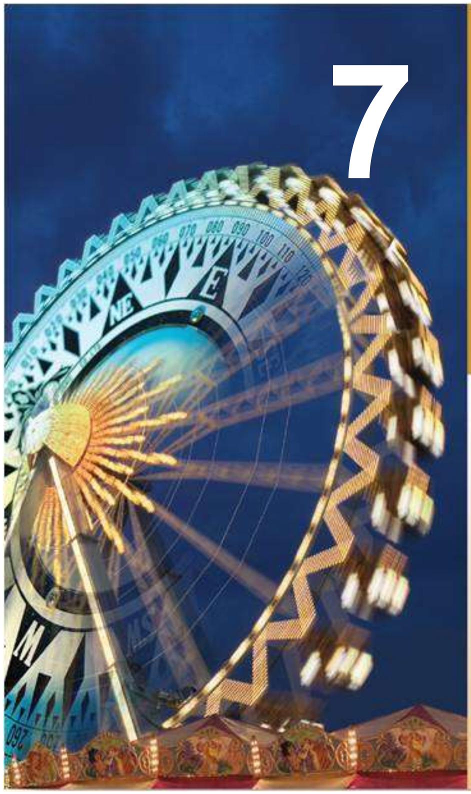

7 Triangles

SECTION 7.6

Vectors: The Dot Product

Learning Objectives

1. Compute a dot product.
2. Find the angle between two vectors.
3. Determine if two nonzero vectors are perpendicular.
4. Use the dot product to calculate work.

Vectors: The Dot Product

Vectors: The Dot Product

Now that we have a way to represent vectors algebraically, we can define a type of multiplication between two vectors.

The *dot product* (also called the *scalar product*) is a form of multiplication that results in a scalar quantity.

For our purposes, it will be useful when finding the angle between two vectors or for finding the work done by a force in moving an object.

Vectors: The Dot Product

Here is the definition of the dot product of two vectors.

DEFINITION

The *dot product* of two vectors $\mathbf{U} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{V} = c\mathbf{i} + d\mathbf{j}$ is written $\mathbf{U} \cdot \mathbf{V}$ and is defined as follows:

$$\begin{aligned} \mathbf{U} \cdot \mathbf{V} &= (a\mathbf{i} + b\mathbf{j}) \cdot (c\mathbf{i} + d\mathbf{j}) \\ &= (ac) + (bd) \end{aligned}$$

As you can see, the dot product is a real number (scalar), not a vector.

Example 1

Find each of the following dot products.

a. $\mathbf{U} \cdot \mathbf{V}$ when $\mathbf{U} = \langle 3, 4 \rangle$ and $\mathbf{V} = \langle 2, 5 \rangle$

b. $\langle -1, 2 \rangle \cdot \langle 3, -5 \rangle$

c. $\mathbf{S} \cdot \mathbf{W}$ when $\mathbf{S} = 6\mathbf{i} + 3\mathbf{j}$ and $\mathbf{W} = 2\mathbf{i} - 7\mathbf{j}$

Solution:

For each problem, we simply multiply the coefficients *a* and *c* and add that result to the product of the coefficients *b* and *d*.

Example 1 – *Solution*

cont'd

a. $\mathbf{U} \cdot \mathbf{V} = 3(2) + 4(5)$

$= 6 + 20$

$= 26$

b. $\langle -1, 2\rangle \cdot \langle 3, -5\rangle = -1(3) + 2(-5)$

$= -3 + (-10)$

$= -13$

c. $\mathbf{S} \cdot \mathbf{W} = 6(2) + 3(-7)$

$= 12 + (-21)$

$= -9$

Finding the Angle Between Two Vectors

Finding the Angle Between Two Vectors

One application of the dot product is finding the angle between two vectors. To do this, we will use an alternate form of the dot product, shown in the following theorem.

THEOREM 7.1

The dot product of two vectors is equal to the product of their magnitudes multiplied by the cosine of the angle between them. That is, when θ is the angle between two nonzero vectors $\mathbf{U}$ and $\mathbf{V}$, then

$$\mathbf{U} \bullet \mathbf{V} = |\mathbf{U}||\mathbf{V}| \cos \theta$$

Finding the Angle Between Two Vectors

When we are given two vectors and asked to find the angle between them, we rewrite the formula in Theorem 7.1 by dividing each side by $|\mathbf{U}||\mathbf{V}|$.

The result is

$$\cos\theta = \frac{\mathbf{U} \cdot \mathbf{V}}{|\mathbf{U}||\mathbf{V}|}$$

This formula is equivalent to our original formula, but is easier to work with when finding the angle between two nonzero vectors.

Example 2

Find the angle between the vectors **U** and **V.**

a. $\mathbf{U} = \langle 2, 3 \rangle$ and $\mathbf{V} = \langle -3, 2 \rangle$ **b.** $\mathbf{U} = 6\mathbf{i} - \mathbf{j}$ and $\mathbf{V} = \mathbf{i} + 4\mathbf{j}$

Solution:

a. $$\cos\theta = \frac{\mathbf{U} \cdot \mathbf{V}}{|\mathbf{U}||\mathbf{V}|}$$

$$= \frac{2(-3) + 3(2)}{\sqrt{2^2 + 3^2} \cdot \sqrt{(-3)^2 + 2^2}}$$

$$= \frac{-6 + 6}{\sqrt{13} \cdot \sqrt{13}}$$

Example 2 – *Solution*

cont'd

$$= \frac{0}{13}$$

$$\cos\theta = 0$$

$$\theta = 90°$$

b. $$\cos\theta = \frac{\mathbf{U} \bullet \mathbf{V}}{|\mathbf{U}||\mathbf{V}|}$$

$$= \frac{6(1) + (-1)4}{\sqrt{6^2 + (-1)^2} \cdot \sqrt{1^2 + 4^2}}$$

$$= \frac{6 + (-4)}{\sqrt{37} \cdot \sqrt{17}}$$

Example 2 – *Solution*

cont'd

$$= \frac{2}{25.08}$$

$$\cos\theta = 0.0797$$

$$\theta = \cos^{-1}(0.0797) = 85.43^\circ$$

To the nearest hundredth of a degree

Perpendicular Vectors

Perpendicular Vectors

If two nonzero vectors are perpendicular, then the angle between them is 90°. We sometimes refer to perpendicular vectors as being orthogonal. Because the cosine of 90° is always 0, the dot product of two perpendicular vectors must also be 0. This fact gives rise to the following theorem.

THEOREM 7.2

If $\mathbf{U}$ and $\mathbf{V}$ are two nonzero vectors, then

$$\mathbf{U} \bullet \mathbf{V} = 0 \Longleftrightarrow \mathbf{U} \perp \mathbf{V}$$

In Words: Two nonzero vectors are perpendicular, or orthogonal, if and only if their dot product is 0.

Example 3

Given vectors $\mathbf{U} = 8\mathbf{i} + 6\mathbf{j}$, $\mathbf{V} = 3\mathbf{i} - 4\mathbf{j}$, and $\mathbf{W} = 4\mathbf{i} + 3\mathbf{j}$, determine if **U** is perpendicular to either **V** or **W.**

Solution:

Find **U** • **V** and **U** • **W**. If the dot product is zero, then the two vectors are perpendicular.

$$\mathbf{U} \cdot \mathbf{V} = 8(3) + 6(-4)$$

$$= 24 - 24$$

$$= 0$$

Therefore, **U** and **V** are perpendicular

Example 3 – *Solution*

cont'd

$$\mathbf{U} \cdot \mathbf{W} = 8(4) + (6)3$$

$$= 32 + 18$$

$$= 50$$

Therefore, **U** and **W** are not perpendicular

Work

Work

We have known that work is performed when a constant force **F** is used to move an object a certain distance. We can represent the movement of the object using a displacement vector, **d**, as shown in Figure 1.

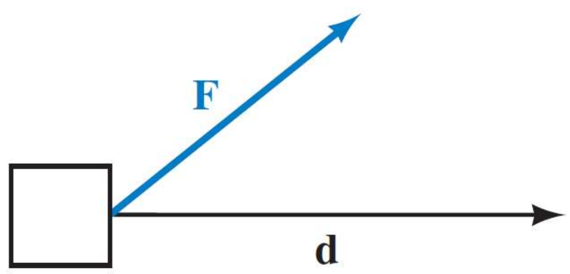

Figure 1

Work

In Figure 2 we let **V** represent the component of **F** that is oriented in the same direction as **d**, since only the amount of the force in the direction of movement can be used in calculating work.

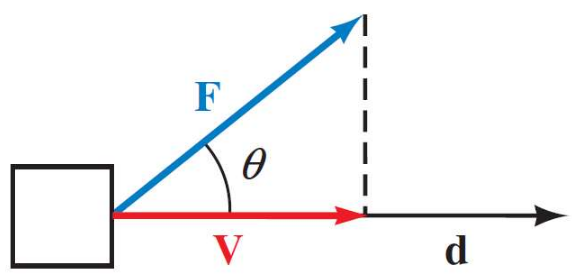

Figure 2

Work

V is sometimes called the *projection of* **F** *onto* **d**. We can find the magnitude of **V** using right triangle trigonometry:

$$\left|\mathbf{V}\right| = \left|\mathbf{F}\right| \cos \theta$$

Because $\left|\mathbf{d}\right|$ represents the distance the object is moved, the work performed by the force is

$$\begin{aligned}
\text{Work} &= \left|\mathbf{V}\right|\left|\mathbf{d}\right| \\
&= (\left|\mathbf{F}\right| \cos \theta) \cdot \left|\mathbf{d}\right| \\
&= \left|\mathbf{F}\right|\left|\mathbf{d}\right| \cos \theta \\
&= \mathbf{F} \bullet \mathbf{d} && \text{By Theorem 7.1}
\end{aligned}$$

Work

We have just established the following theorem.

THEOREM 7.3

If a constant force $\mathbf{F}$ is applied to an object, and the resulting movement of the object is represented by the displacement vector $\mathbf{d}$, then the work performed by the force is

$$\text{Work} = \mathbf{F} \cdot \mathbf{d}$$

Example 4

A force $\mathbf{F} = 35\mathbf{i} - 12\mathbf{j}$ (in pounds) is used to push an object up a ramp. The resulting movement of the object is represented by the displacement vector $\mathbf{d} = 15\mathbf{i} + 4\mathbf{j}$ (in feet), as illustrated in Figure 3. Find the work done by the force.

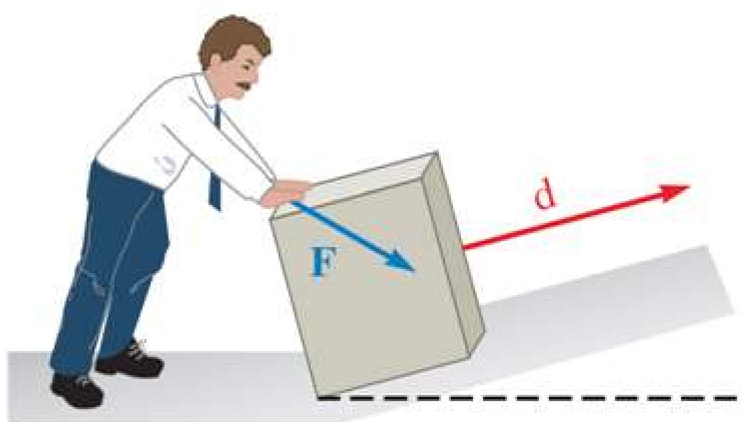

Figure 3

Example 4 – *Solution*

By Theorem 7.3,

$$\text{Work} = \mathbf{F} \cdot \mathbf{d}$$

$$= 35(15) + (-12)(4)$$

$$= 480 \text{ ft-lb} \qquad \text{To two significant digits}$$