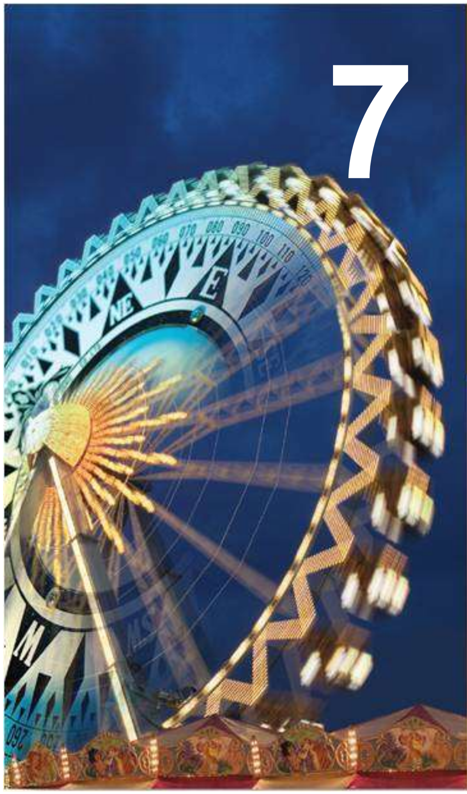

7 Triangles

SECTION 7.4 The Area of a Triangle

Learning Objectives

1. Calculate the area of a triangle given two sides and the included angle.

2. Calculate the area of a triangle given two angles and one side.

3. Find the semiperimeter of a triangle.

4. Use Heron's formula to find the area of a triangle.

Two Sides and the Included Angle

Two Sides and the Included Angle

To derive our first formula, we begin with the general formula for the area of a triangle:

$$S = \frac{1}{2}(\text{base})(\text{height})$$

The base of triangle *ABC* in Figure 1 is *c* and the height is *h*. So the formula for *S* becomes, in this case,

$$S = \frac{1}{2}ch$$

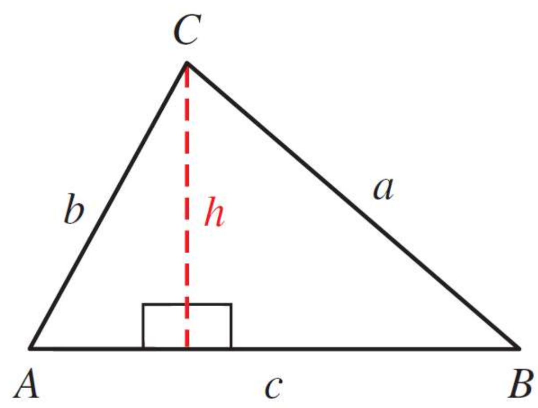

Figure 1

Two Sides and the Included Angle

Suppose that, for triangle *ABC*, we are given the lengths of sides *b* and *c* and the measure of angle *A*. Then we can write sin *A* as

$$\sin A = \frac{h}{b}$$

or, by solving for *h*,

$$h = b \sin A$$

Substituting this expression for *h* into the formula

$$S = \frac{1}{2}ch$$

we have

$$S = \frac{1}{2}bc \sin A$$

Two Sides and the Included Angle

Applying the same kind of reasoning to the heights drawn from *A* and *B*, we also have the following.

AREA OF A TRIANGLE (SAS)

$$S = \frac{1}{2}bc \sin A \qquad S = \frac{1}{2}ac \sin B \qquad S = \frac{1}{2}ab \sin C$$

Each of these three formulas indicates that to find the area of a triangle for which we are given two sides and the angle included between them, we multiply half the product of the two sides by the sine of the angle included between them.

Example 1

Find the area of triangle *ABC* if $A = 35.1°$, $b = 2.43$ cm, and $c = 3.57$ cm.

Solution:

Applying the first formula we derived, we have

$$S = \frac{1}{2} bc \sin A$$

$$= \frac{1}{2}(2.43)(3.57) \sin 35.1°$$

$$= 2.49 \text{ cm}^2$$ To three significant digits

Two Angles and One Side

Two Angles and One Side

Suppose we were given angles *A* and *B* and side *a* in triangle *ABC* in Figure 1.

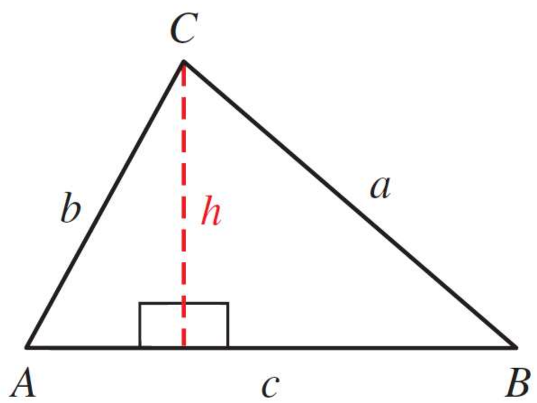

Figure 1

To find side *b*, we use the law of sines $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

Solving this equation for *b* would give us $b = \dfrac{a \sin B}{\sin A}$

Two Angles and One Side

Substituting this expression for *b* into the formula

$$S = \frac{1}{2}ab \sin C$$

we have

$$S = \frac{1}{2}a\left(\frac{a \sin B}{\sin A}\right)\sin C$$

$$= \frac{a^2 \sin B \sin C}{2 \sin A}$$

Two Angles and One Side

A similar sequence of steps can be used to derive the following.

The formula we use depends on the side we are given.

AREA OF A TRIANGLE (AAS)

$$S = \frac{a^2 \sin B \sin C}{2 \sin A} \qquad S = \frac{b^2 \sin A \sin C}{2 \sin B} \qquad S = \frac{c^2 \sin A \sin B}{2 \sin C}$$

Example 2

Find the area of triangle *ABC* given $A = 24^\circ\ 10'$, $B = 120^\circ\ 40'$, and $a = 4.25$ ft.

Solution:

We begin by finding *C*.

$$\begin{aligned} C &= 180^\circ - (24^\circ\ 10' + 120^\circ\ 40') \\ &= 35^\circ\ 10' \end{aligned}$$

Example 2 – *Solution*

cont'd

Now, applying the formula

$$S = \frac{a^2 \sin B \sin C}{2 \sin A}$$

with $a = 4.25$, $A = 24° \ 10'$, $B = 120° \ 40'$, and $C = 35° \ 10'$, we have

$$S = \frac{(4.25)^2(\sin 120° \ 40')(\sin 35° \ 10')}{2 \sin 24° \ 10'}$$

$$= \frac{(4.25)^2(0.8601)(0.5760)}{2(0.4094)}$$

$$= 10.9 \text{ ft}^2 \qquad \text{To three significant digits}$$

Three Sides

Three Sides

The last area formula we will discuss is called *Heron's formula*. It is used to find the area of a triangle in which all three sides are known.

AREA OF A TRIANGLE (SSS)

Given a triangle with sides of length a, b, and c, the *semiperimeter* of the triangle is defined as

$$s = \frac{1}{2}(a + b + c)$$

and the area of the triangle is given by

$$S = \sqrt{s(s - a)(s - b)(s - c)}$$

The semiperimeter is simply half the perimeter of the triangle.

Example 3

Find the area of triangle *ABC* if *a* = 12 m, *b* = 14 m, and *c* = 8.0 m.

Solution:

We begin by calculating the formula for *s*, half the perimeter of *ABC*.

$$s = \frac{1}{2}(12 + 14 + 8)$$

$$= 17$$

Example 3 – *Solution*

cont'd

Substituting this value of *s* into Heron's formula along with the given values of *a*, *b*, and *c*, we have

$$S = \sqrt{17(17 - 12)(17 - 14)(17 - 8)}$$

$$= \sqrt{17(5)(3)(9)}$$

$$= \sqrt{2,295}$$

$$= 48 \text{ m}^2 \qquad \text{To two significant digits}$$