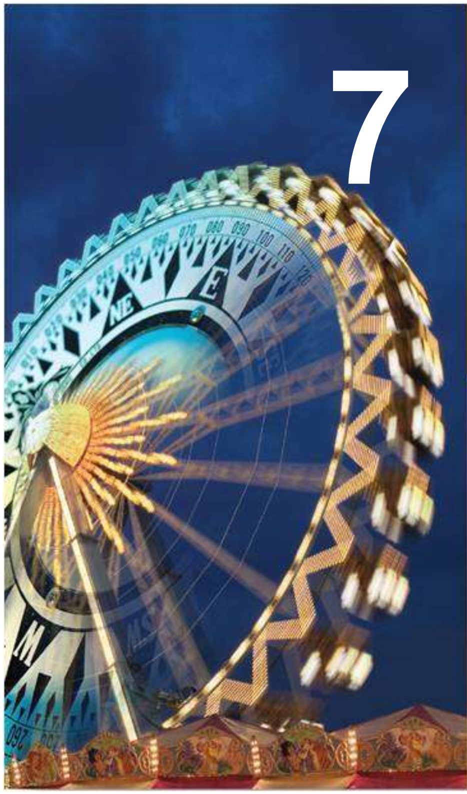


7 Triangles

SECTION 7.3

The Ambiguous Case

Learning Objectives

1. Use the law of sines to find all possible solutions for the ambiguous case.
2. Use the law of cosines to find all possible solutions for the ambiguous case.
3. Solve applied problems involving the ambiguous case.

Using the Law of Sines

Using the Law of Sines

There are different possibilities that arise in solving a triangle for which we are given two sides and an angle opposite one of the given sides. We call this situation as the *ambiguous case*.

Example 2

Find the missing parts in triangle *ABC* if *a* = 54 cm, *b* = 62 cm, and *A* = 40°.

Solution:

First we solve for sin *B* with the law of sines.

Angle B

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b \sin A}{a}$$

$$= \frac{62 \sin 40^\circ}{54} = 0.7380$$

Example 2 – *Solution*

cont'd

Now, because sin *B* is positive for any angle in QI or QII, we have two possibilities (Figure 6).

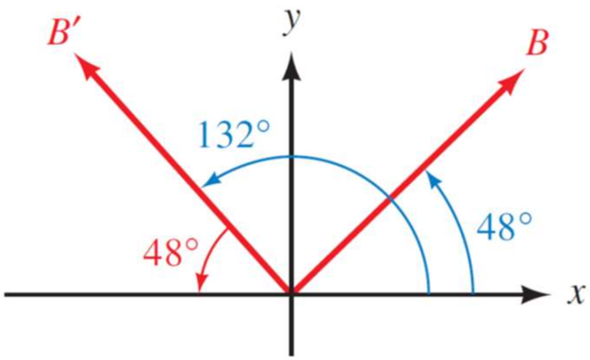


Figure 6

We will call one of them *B* and the other *B*′.

$$B = \sin^{-1}(0.7380) = 48^\circ \qquad \text{or} \qquad B' = 180^\circ - 48^\circ = 132^\circ$$

Example 2 – *Solution*

cont'd

Notice that $B' = 132°$ is the supplement of B. We have two different angles that can be found with $a = 54$ cm, $b = 62$ cm, and $A = 40°$.

Figure 7 shows both of them.

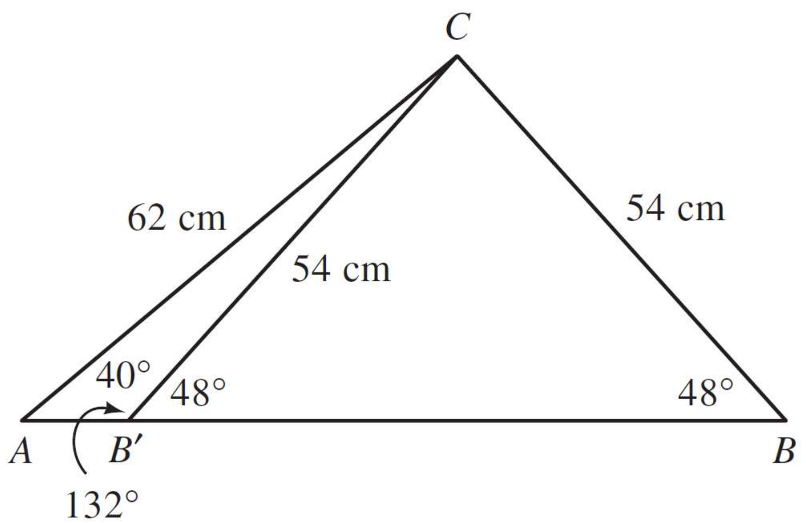


Figure 7

Example 2 – *Solution*

cont'd

One is labeled ABC, while the other is labeled $AB'C$. Also, because triangle $B'BC$ is isosceles, we can see that angle $AB'C$ must be supplementary to B.

Angles C and C'

There are two values for B, so we have two values for C.

$$C = 180 - (A + B) \qquad \text{and} \qquad C' = 180 - (A + B')$$

$$= 180 - (40° + 48°) \qquad\qquad = 180 - (40° + 132°)$$

$$= 92° \qquad\qquad = 8°$$

Example 2 – *Solution*

cont'd

Sides c and c′

$$c = \frac{a \sin C}{\sin A} \qquad \text{and} \qquad c' = \frac{a \sin C'}{\sin A}$$

$$= \frac{54 \sin 92°}{\sin 40°} \qquad\qquad = \frac{54 \sin 8°}{\sin 40°}$$

$$= 84 \text{ cm} \qquad\qquad = 12 \text{ cm}$$

To two significant digits

Example 2 – *Solution*

cont'd

Figure 8 shows both triangles.

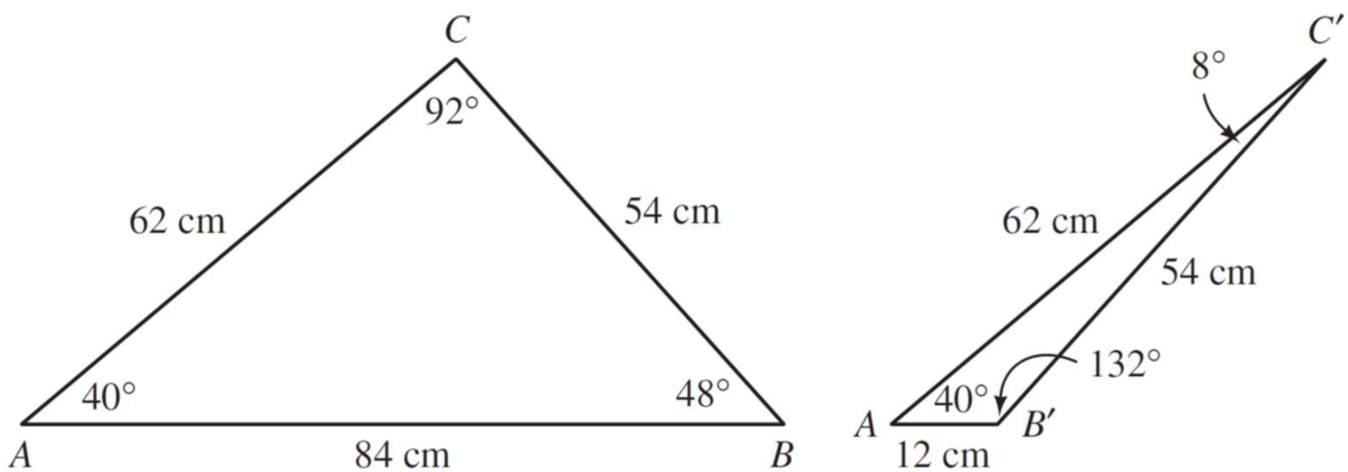


Figure 8

Using the Law of Sines

The following table completes the set of conditions under which we will have 1, 2, or no triangles in the ambiguous case. In Table 1, we are assuming that we are given angle *A* and sides *a* and *b* in triangle *ABC*, and that *h* is the altitude from vertex *C*.

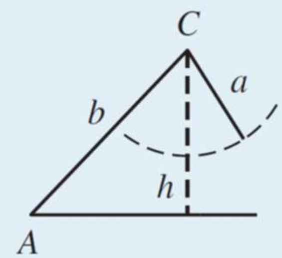
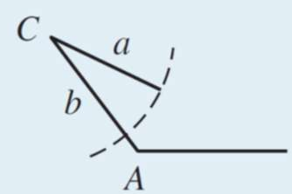
Conditions	Number of Triangles	Diagram
$A < 90^\circ$ and $a < h$	0	
$A > 90^\circ$ and $a < b$	0	

Table 1

Using the Law of Sines

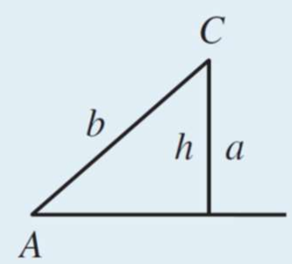
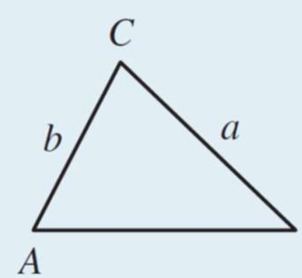
Conditions	Number of Triangles	Diagram
$A < 90°$ and $a = h$	1	
$A < 90°$ and $a \geq b$	1	

Table 1 (continued)

Using the Law of Sines

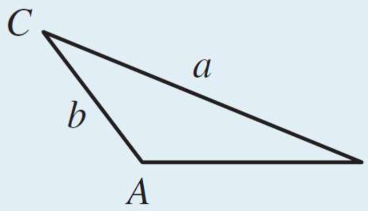
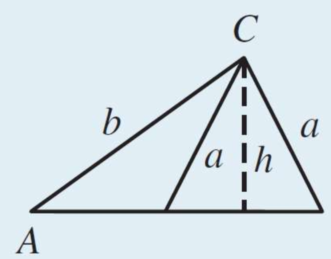
Conditions	Number of Triangles	Diagram
$A > 90°$ and $a > b$	1	
$A < 90°$ and $h < a < b$	2	

Table 1 (continued)

Using the Law of Cosines

Using the Law of Cosines

The ambiguous case can also be solved using the law of cosines.

Example 5

Find the missing parts in triangle *ABC* if $a = 54$ cm, $b = 62$ cm, and $A = 40°$.

Solution:

Once again, we begin by using the law of cosines to find side *c*.

Side c

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$54^2 = 62^2 + c^2 - 2 \cdot 62 \cdot c \cos 40°$$

$$2{,}916 = 3{,}844 + c^2 - 124c(0.7660) \qquad \text{Approximate } \cos 40°$$

$$0 = c^2 - 94.984c + 928$$

Example 5 – *Solution*

cont'd

Using the quadratic formula to solve for *c*, we obtain

$$c = \frac{-(-94.984) \pm \sqrt{(-94.984)^2 - 4(1)(928)}}{2(1)}$$

$$= \frac{94.984 \pm 72.8695}{2}$$

which gives us $c = 11$ or $c = 84$, to two significant digits.

Since both of these values are positive real numbers, there are two triangles possible. We can use the law of cosines again to find either of the remaining two angles.

Example 5 – *Solution*

cont'd

Angles C and C′

$c = 11$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{54^2 + 62^2 - 11^2}{2(54)(62)}$$

$$= 0.9915$$

So $C = \cos^{-1}(0.9915)$

$$= 7°$$

and

$c' = 84$

$$\cos C' = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{54^2 + 62^2 - 84^2}{2(54)(62)}$$

$$= -0.0442$$

So $C' = \cos^{-1}(-0.0442)$

$$= 93°$$

Example 5 – *Solution*

cont'd

Angles B and B′

$$B = 180^\circ - 40^\circ - 7^\circ \qquad \text{and} \qquad B' = 180^\circ - 40^\circ - 93^\circ$$

$$= 133^\circ \qquad\qquad\qquad\qquad = 47^\circ$$

Applications

Example 7

A plane is flying with an airspeed of 170 miles per hour and a heading of 52.5°. Its true course, however, is at 64.1° from due north. If the wind currents are a constant 40.0 miles per hour, what are the possibilities for the ground speed of the plane?

Solution:

We represent the velocity of the plane and the velocity of the wind with vectors, the resultant of which will be the ground speed and true course of the plane.

Example 7 – *Solution*

cont'd

First, measure an angle of 52.5° from north (clockwise) and draw a vector in that direction with magnitude 170 to represent the heading and airspeed.

Then draw a second vector at an angle of 64.1° from north with unknown magnitude to represent the true course and ground speed.

Because this second vector must be the resultant, the wind velocity will be some vector originating at the tip of the first vector and ending at the tip of the second vector, and having a length of 40.0.

Example 7 – *Solution*

cont'd

Figure 10 illustrates the situation. From Figure 10, we see that

$$\theta = 64.1^\circ - 52.5^\circ = 11.6^\circ$$

First we use the law of sines to find angle α.

$$\frac{\sin \alpha}{170} = \frac{\sin 11.6^\circ}{40.0}$$

$$\sin \alpha = \frac{170 \sin 11.6^\circ}{40.0}$$

$$= 0.8546$$

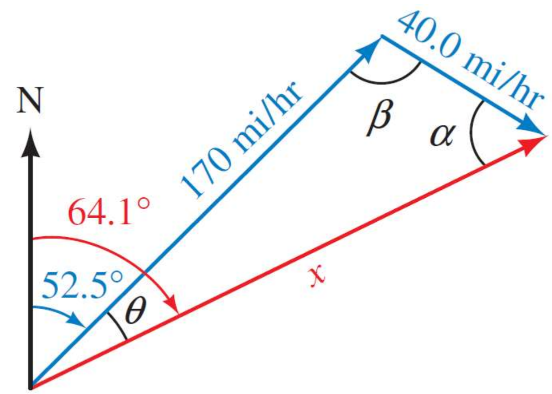


Figure 10

Example 7 – *Solution*

cont'd

Because sin α is positive in quadrants I and II, we have two possible values for α.

$$\alpha = \sin^{-1}(0.8546) = 58.7° \qquad \text{and} \qquad \alpha' = 180° - 58.7°$$

$$= 121.3°$$

We see there are two different triangles that can be formed with the three vectors.

Example 7 – *Solution*

cont'd

Figure 11 shows both of them.

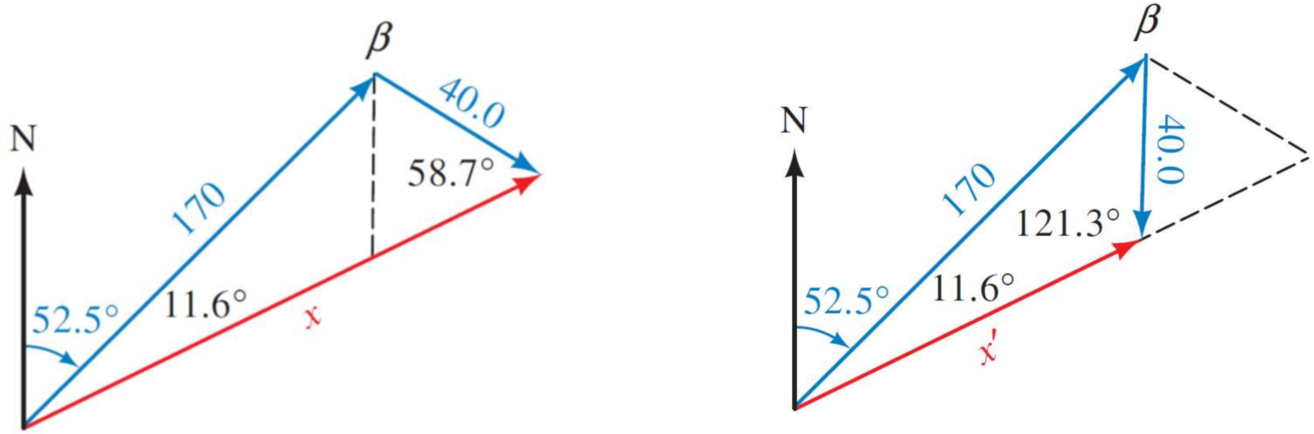


Figure 11

There are two values for α, so there are two values for β.

$$\beta = 180° - (11.6° + 58.7°) \quad \text{and} \quad \beta' = 180° - (11.6° + 121.3°)$$

$$= 109.7° \qquad\qquad = 47.1°$$

Example 7 – *Solution*

cont'd

Now we can use the law of sines again to find the ground speed.

$$\frac{x}{\sin 109.7°} = \frac{40}{\sin 11.6°} \quad \text{and} \quad \frac{x'}{\sin 47.1°} = \frac{40}{\sin 11.6°}$$

$$x = \frac{40 \sin 109.7°}{\sin 11.6°} \qquad\qquad x' = \frac{40 \sin 47.1°}{\sin 11.6°}$$

$$= 187 \text{ mi/hr} \qquad\qquad = 146 \text{ mi/hr}$$

To three significant digits, the two possibilities for the ground speed of the plane are 187 miles per hour and 146 miles per hour.