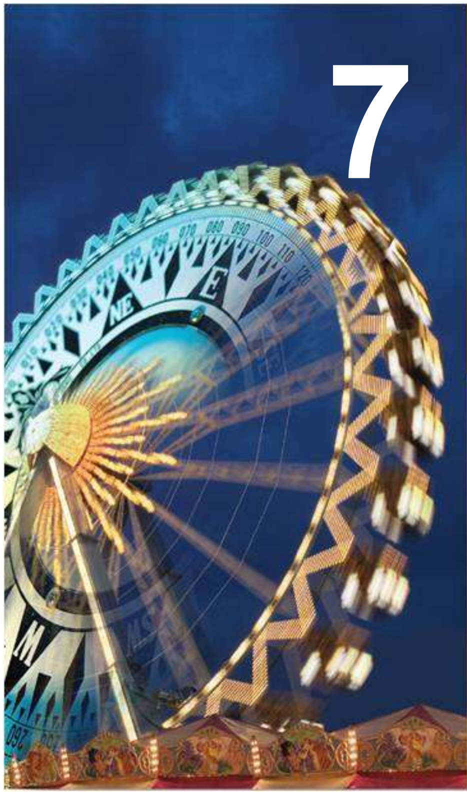

Triangles

SECTION 7.2

The Law of Cosines

Learning Objectives

1 Use the law of cosines to find a missing side in an oblique triangle.

2 Use the law of cosines to find a missing angle in an oblique triangle.

3 Draw a vector representing a given heading.

4 Use the law of cosines to solve a real-life problem involving heading or true course.

The Law of Cosines

In this section, we will derive another relationship that exists between the sides and angles in any triangle. It is called the *law of cosines* and is stated like this:

LAW OF COSINES (SAS)

Given triangle ABC shown in Figure 1:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

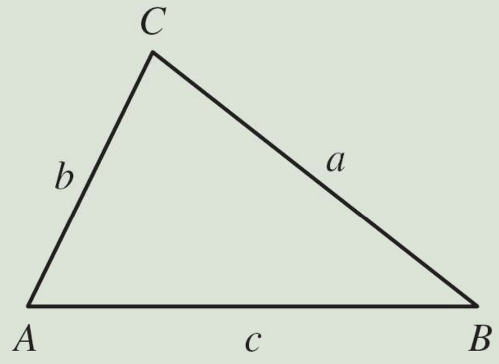

Figure 1

Derivation

Derivation

To derive the formulas stated in the law of cosines, we apply the Pythagorean Theorem and some of our basic trigonometric identities. Applying the Pythagorean Theorem to right triangle *BCD* in Figure 2, we have

$$\begin{aligned} a^2 &= (c - x)^2 + h^2 \\ &= c^2 - 2cx + x^2 + h^2 \end{aligned}$$

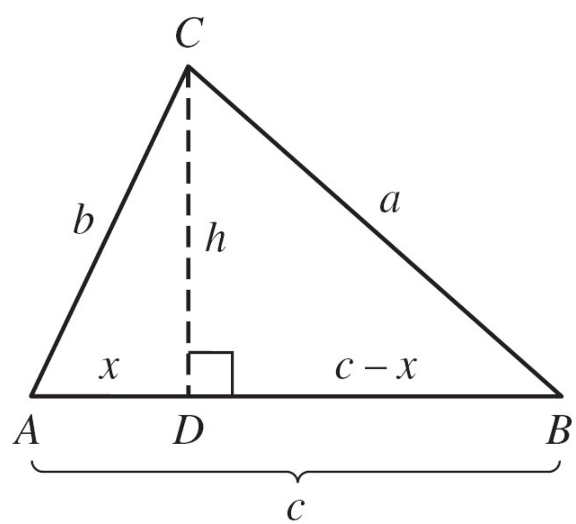

Figure 2

Derivation

But from right triangle *ACD*, we have $x^2 + h^2 = b^2$, so

$$a^2 = c^2 - 2cx + b^2$$

$$= b^2 + c^2 - 2cx$$

Now, since cos *A* = *x*/*b*, we have *x* = *b* cos *A*, or

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Applying the same sequence of substitutions and reasoning to the right triangles formed by the altitudes from vertices *A* and *B* will give us the other two formulas listed in the law of cosines.

Derivation

We can use the law of cosines to solve triangles for which we are given two sides and the angle included between them (SAS) or triangles for which we are given all three sides (SSS).

Two Sides and the Included Angle

Example 1

Find the missing parts of triangle *ABC* if $A = 60°$, $b = 25$ inches, and $c = 32$ inches.

Solution:

A diagram of the given information is shown in Figure 3.

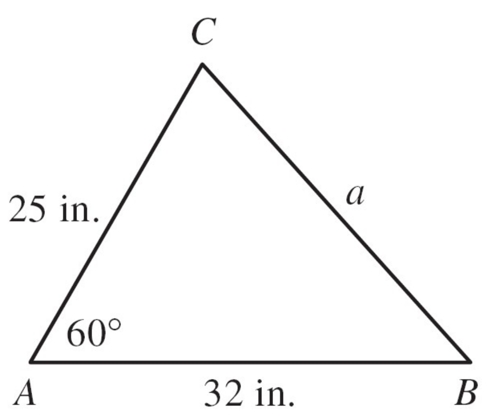

Figure 3

Example 1 – *Solution*

cont'd

The solution process will include the use of both the law of cosines and the law of sines. We begin by using the law of cosines to find *a*.

Side a

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad \text{Law of cosines}$$

$$= 25^2 + 32^2 - 2(25)(32) \cos 60° \qquad \text{Substitute in given values}$$

$$= 625 + 1{,}024 - 1{,}600(0.5) \qquad \text{Calculator}$$

$$a^2 = 849$$

$$a = 29 \text{ inches} \qquad \text{To two significant digits}$$

Example 1 – *Solution*

cont'd

Now that we have *a*, we can use either the law of sines or the law of cosines to solve for angle *B* or *C*.

When we have a choice of angles to solve for, and we are using the law of sines to do so, it is best to solve for the smaller angle.

This is because a triangle can have at most one obtuse angle, which, if present, must be opposite the longest side. Since *c* is the longest side, only angle *C* might be obtuse.

Therefore we solve for angle *B* first because we know it must be acute. If using the law of cosines, it does not matter which angle we solve for first.

Example 1 – *Solution*

cont'd

Angle B

Using the law of sines

$$\sin B = \frac{b \sin A}{a} = \frac{25 \sin 60°}{29} = 0.7466$$

So

$$B = \sin^{-1}(0.7466) = 48° \qquad \text{To the nearest degree}$$

Example 1 – *Solution*

cont'd

Angle B

Using the law of cosines

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$25^2 = 29^2 + 32^2 - 2(29)(32)\cos B$$

$$625 = 1{,}865 - 1{,}856\cos B$$

$$-1{,}240 = -1{,}856\cos B$$

$$0.6681 = \cos B$$

So

$$B = \cos^{-1}(0.6681) = 48^\circ$$ To the nearest degree

Example 1 – *Solution*

cont'd

Angle C

$$C = 180° - (A + B)$$

$$= 180° - (60° + 48°)$$

$$= 72°$$

Three Sides

Three Sides

To use the law of cosines to solve a triangle for which we are given all three sides, it is convenient to rewrite the equations with the cosines isolated on one side.

Here is an equivalent form of the law of cosines. The first formula is the one we just derived.

LAW OF COSINES (SSS)

Given triangle *ABC* shown in Figure 5,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

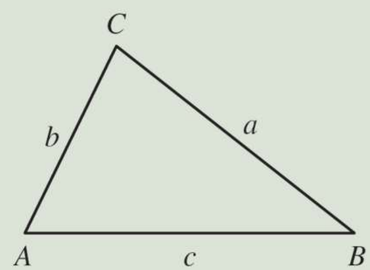

Figure 5

Navigation

Navigation

DEFINITION

The *heading* of an object is the angle, measured clockwise from due north, to the vector representing the *intended* path of the object.

DEFINITION

The *true course* of an object is the angle, measured clockwise from due north, to the vector representing the *actual path* of the object.

With the specific case of objects in flight, we use the term *airspeed* for the speed of the object relative to the air and *ground speed* for the speed of the object relative to the ground.

Navigation

The airspeed is the magnitude of the vector representing the velocity of the object in the direction of the heading, and the ground speed is the magnitude of the vector representing the velocity of the object in the direction of the true course.

Likewise, the *wind speed* is the magnitude of the vector representing the wind.

Navigation

As illustrated in Figure 7, the ground speed/true course vector is the resultant vector found by adding the airspeed/heading vector and the wind speed/wind direction vector.

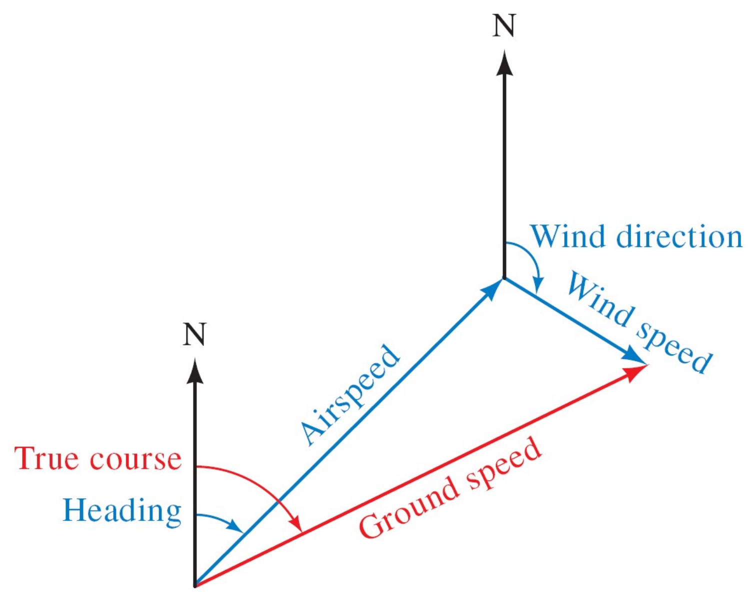

Figure 7

Example 4

A plane is flying with an airspeed of 185 miles per hour with heading 120°. The wind currents are running at a constant 32 miles per hour at 165° clockwise from due north. Find the true course and ground speed of the plane.

Solution:

Figure 8 is a diagram of the situation with the vector **V** representing the airspeed and direction of the plane and **W** representing the speed and direction of the wind currents.

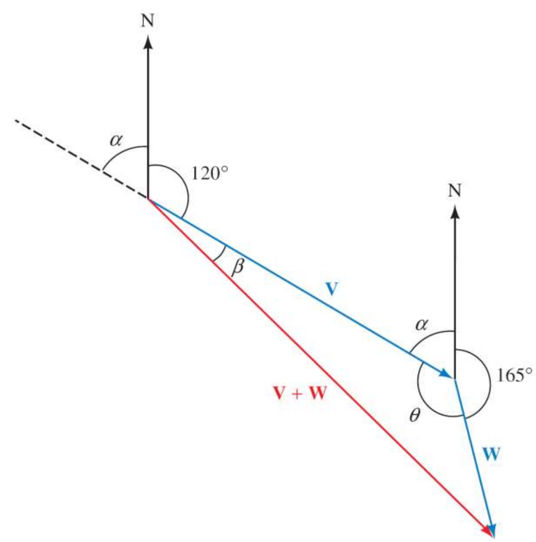

Figure 8

Example 4 – *Solution*

cont'd

From Figure 8, we see that

$$\alpha = 180° - 120° = 60°$$

and

$$\theta = 360° - (\alpha + 165°) = 135°$$

We now have the case SAS. The magnitude of **V** + **W** can be found from the law of cosines.

Example 4 – *Solution*

cont'd

$$|\mathbf{V} + \mathbf{W}|^2 = |\mathbf{V}|^2 + |\mathbf{W}|^2 - 2|\mathbf{V}||\mathbf{W}|\cos\theta$$

$$= 185^2 + 32^2 - 2(185)(32)\cos 135°$$

$$= 43{,}621$$

so $|\mathbf{V} + \mathbf{W}| = 210$ mph   To two significant digits

To find the direction of **V + W**, we first find β using the law of sines.

$$\frac{\sin\beta}{32} = \frac{\sin\theta}{210}$$

Example 4 – *Solution*

cont'd

$$\sin \beta = \frac{32 \sin 135°}{210}$$

$$= 0.1077$$

so $\quad \beta = \sin^{-1}(0.1077) = 6°$ $\quad$ To the nearest degree

The true course is 120° + β = 120° + 6° = 126°. The speed of the plane with respect to the ground is 210 mph.