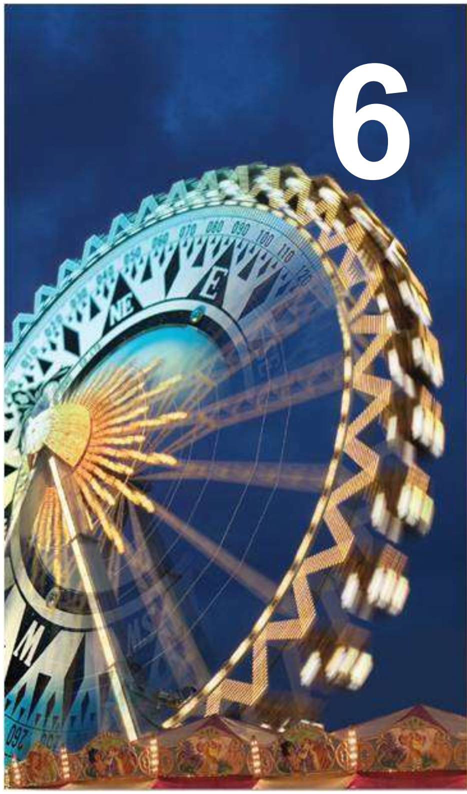


6 Equations

SECTION 6.4

Parametric Equations and Further Graphing

Learning Objectives

1. Graph a plane curve by plotting points.
2. Indicate the orientation of a plane curve.
3. Eliminate the parameter from a pair of parametric equations.
4. Use parametric equations as a model in a real-life problem.

Parametric Equations and Further Graphing

Let's begin with a model of the giant wheel built by George Ferris. The diameter of this wheel is 250 feet, and the bottom of the wheel sits 14 feet above the ground.

We can superimpose a coordinate system on our model, so that the origin of the coordinate system is at the center of the wheel.
This is shown in Figure 1.

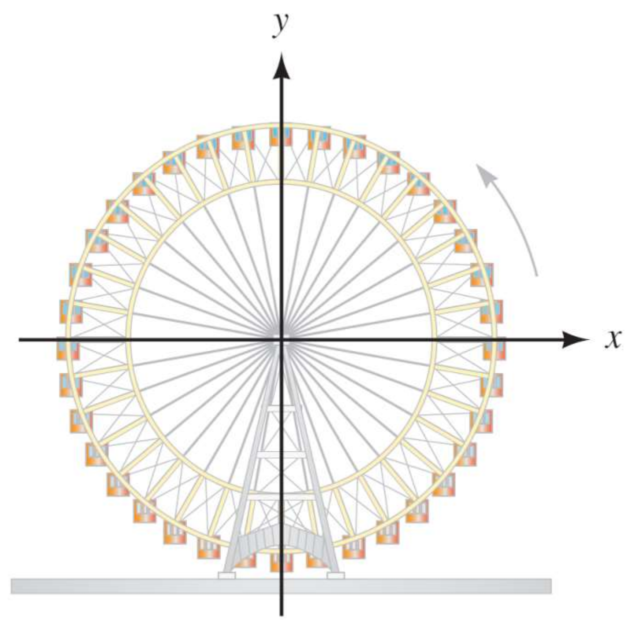


Figure 1

Choosing one of the carriages in the first quadrant as the point (x, y), we draw a line from the origin to the carriage. Then we draw the right triangle shown in Figure 2.

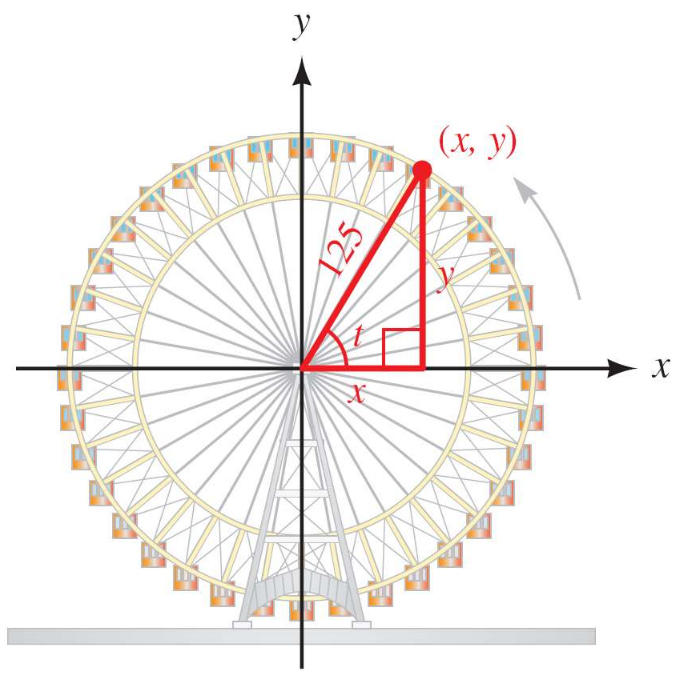


Figure 2

Parametric Equations and Further Graphing

From the triangle we have

$$\cos t = \frac{x}{125} \Rightarrow x = 125 \cos t$$

$$\sin t = \frac{y}{125} \Rightarrow y = 125 \sin t$$

The two equations on the right are called *parametric equations*. They show x and y as functions of a third variable, t, called the *parameter*. Each number we substitute for t gives us an ordered pair (x, y).

If we graph each of these ordered pairs on a rectangular coordinate system, the resulting curve is called a *plane curve*. We demonstrate this in the next example.

Example 1

Graph the plane curve defined by the parametric equations $x = 125 \cos t$ and $y = 125 \sin t$.

Solution:

We can find points on the graph by choosing values of t and using the two equations to find corresponding values of x and y.

Table 1 shows a number of ordered pairs, which we obtained by letting t be different multiples of $\pi/4$.

t	$x = 125 \cos t$	$y = 125 \sin t$	(x, y)
0	125	0	(125, 0)
$\pi/4$	88	88	(88, 88)
$\pi/2$	0	125	(0, 125)
$3\pi/4$	-88	88	$(-88, 88)$
π	-125	0	$(-125, 0)$
$5\pi/4$	-88	-88	$(-88, -88)$
$3\pi/2$	0	-125	$(0, -125)$
$7\pi/4$	88	-88	$(88, -88)$
2π	125	0	(125, 0)

Table 1

Example 1 – *Solution*

cont'd

The graph of the plane curve is shown in Figure 3.

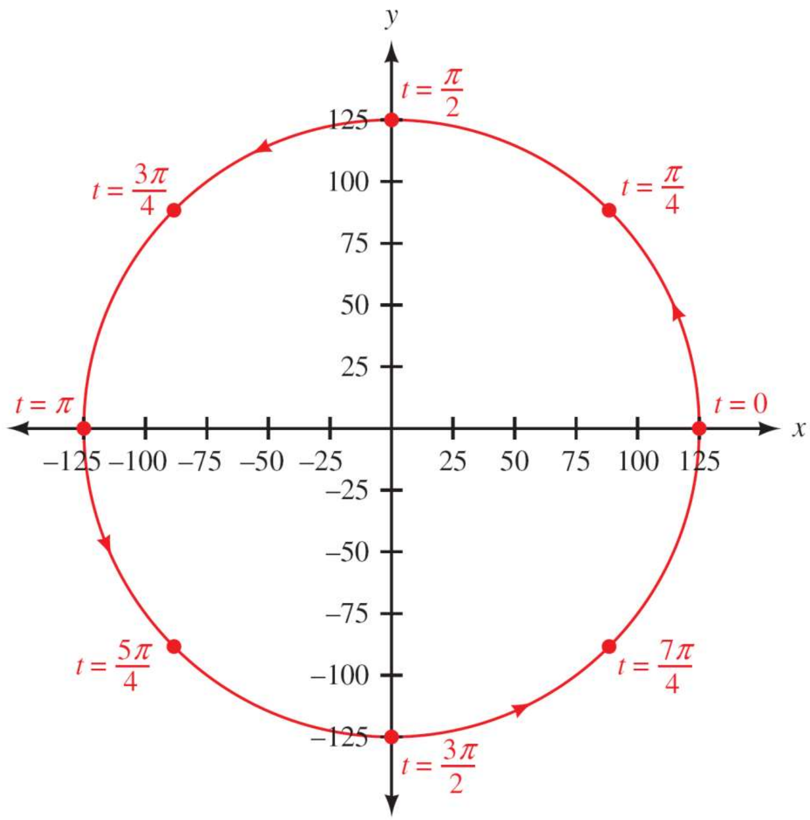


Figure 3

Example 1 – *Solution*

cont'd

We have labeled each point with its associated value of the parameter t. As expected, the curve is a circle.

More importantly, notice in which direction the circle is traversed as t increases from 0 to 2π.

We say that the plane curve is *oriented* in a counterclockwise direction and indicate the orientation using small arrows.

Parametric Equations and Further Graphing

In general, the orientation of a plane curve is the direction the curve is traversed as the parameter increases.

The ability of parametric equations to associate points on a curve with values of a parameter makes them especially suited for describing the path of an object in motion.

Eliminating the Parameter

Eliminating the Parameter

Let's go back to our original set of parametric equations and solve for cos *t* and sin *t*:

$$x = 125 \cos t \;\Rightarrow\; \cos t = \frac{x}{125}$$

$$y = 125 \sin t \;\Rightarrow\; \sin t = \frac{y}{125}$$

Substituting the expressions above for cos *t* and sin *t* into the Pythagorean identity $\cos^2 t + \sin^2 t = 1$, we have

$$\left(\frac{x}{125}\right)^2 + \left(\frac{y}{125}\right)^2 = 1$$

$$\frac{x^2}{125^2} + \frac{y^2}{125^2} = 1$$

$$x^2 + y^2 = 125^2$$

Eliminating the Parameter

We recognize this last equation as the equation of a circle with a radius of 125 and center at the origin.

What we have done is eliminate the parameter t to obtain an equation in just x and y whose graph we recognize.

This process is called *eliminating the parameter*. Note that it gives us further justification that the graph of our set of parametric equations is a circle.

Example 2

Eliminate the parameter t from the parametric equations $x = 3 \cos t$ and $y = 2 \sin t$.

Solution:

Again, we will use the identity $\cos^2 t + \sin^2 t = 1$. Before we do so, however, we must solve the first equation for $\cos t$ and the second equation for $\sin t$.

$$x = 3 \cos t \;\Rightarrow\; \cos t = \frac{x}{3}$$

$$y = 2 \sin t \;\Rightarrow\; \sin t = \frac{y}{2}$$

Example 2 – *Solution*

cont'd

Substituting $x/3$ and $y/2$ for cos t and sin t into the first Pythagorean identity gives us

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

which is the equation of an ellipse.

The center is at the origin, the x-intercepts are 3 and –3, and the y-intercepts are 2 and –2.

Example 2 – *Solution*

cont'd

Figure 8 shows the graph.

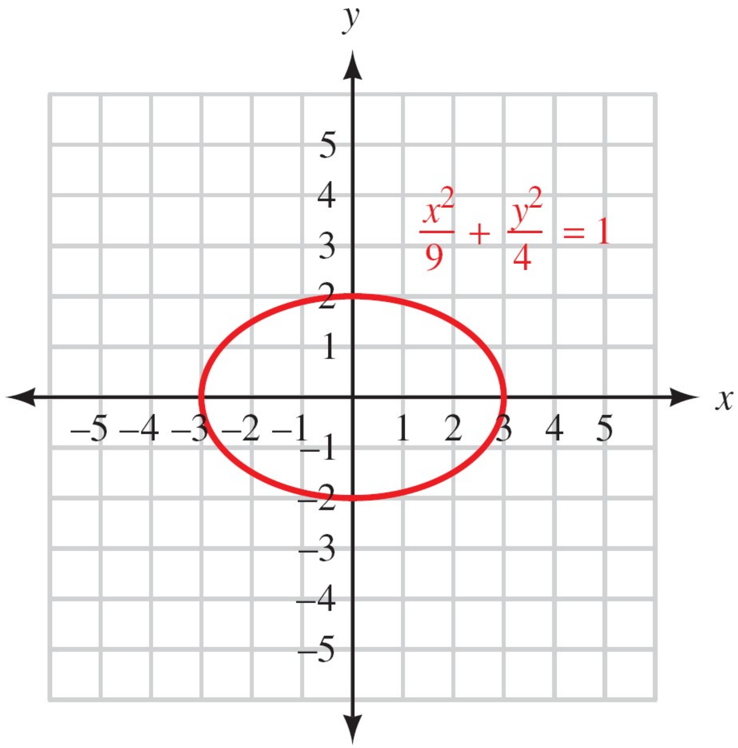


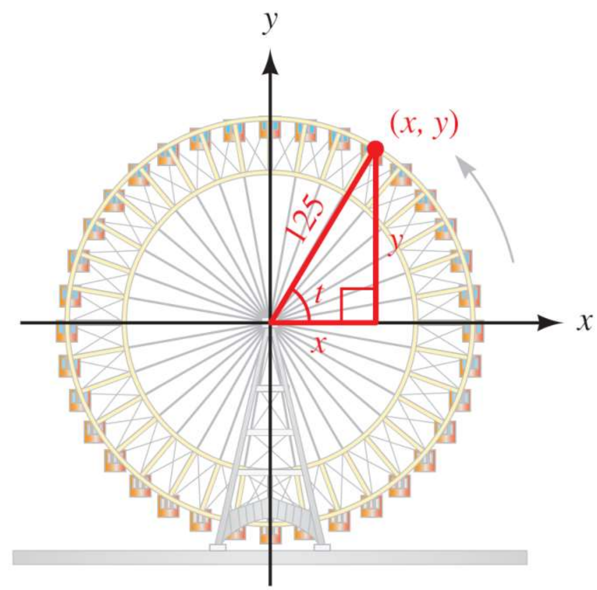
Figure 8

Making Our Models More Realistic

Making Our Models More Realistic

Let's go back to our Ferris wheel model from the beginning of this section. Our wheel has a radius of 125 feet and sits 14 feet above the ground. One trip around the wheel takes 20 minutes.

Figure 11 shows this model with a coordinate system superimposed with its origin at the center of the wheel.



$x = 125 \cos t$
$y = 125 \sin t$

Figure 11

Making Our Models More Realistic

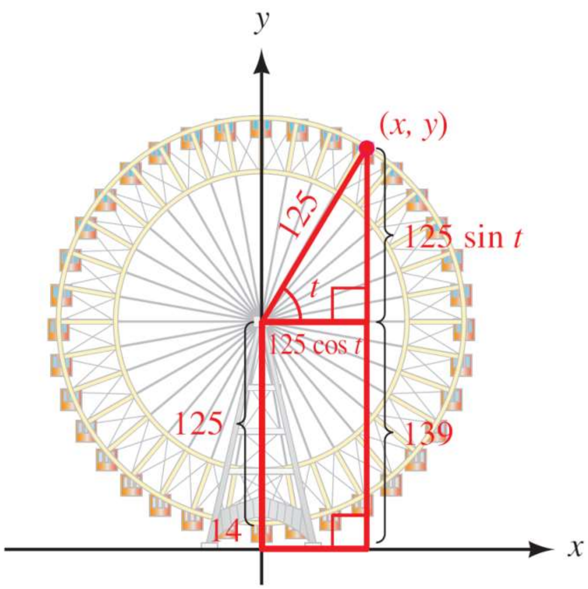
Also shown are the parametric equations that describe the path of someone riding the wheel.

Our model would be more realistic if the x-axis was along the ground, below the wheel.

We can accomplish this very easily by moving everything up 139 feet (125 feet for the radius and another 14 feet for the wheel's height above the ground).

Making Our Models More Realistic

Figure 12 shows our new graph next to a new set of parametric equations.



$x = 125 \cos t$
$y = 139 + 125 \sin t$

Figure 12

Making Our Models More Realistic

Next, let's assume there is a ticket booth 200 feet to the left of the wheel. If we want to use the ticket booth as our starting point, we can move our graph to the right 200 feet.

Figure 13 shows this model, along with the corresponding set of parametric equations.

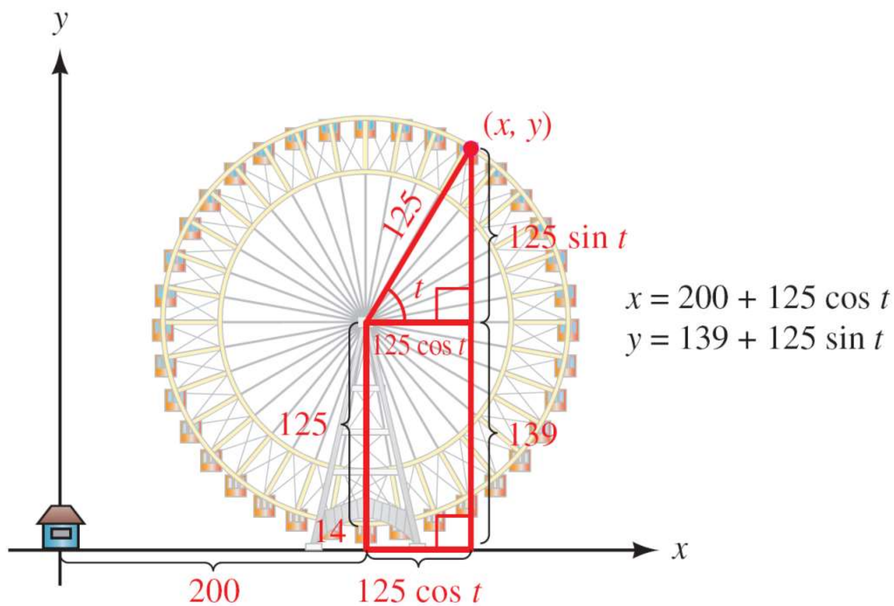


Figure 13

Making Our Models More Realistic

Note:
We can solve this last set of equations for cos *t* and sin *t* to get

$$\cos t = \frac{x - 200}{125} \qquad \sin t = \frac{y - 139}{125}$$

Using these results in our Pythagorean identity, $\cos^2 t + \sin^2 t = 1$, we have $(x - 200)^2 + (y - 139)^2 = 125^2$ which we recognize as the equation of a circle with center at (200, 139) and radius 125.

Making Our Models More Realistic

Continuing to improve our model, we would like the rider on the wheel to start their ride at the bottom of the wheel.

Assuming *t* is in radians, we accomplish this by subtracting $\pi/2$ from *t* giving us

$$x = 200 + 125\cos\left(t - \frac{\pi}{2}\right)$$

$$y = 139 + 125\sin\left(t - \frac{\pi}{2}\right)$$

Finally, one trip around the wheel takes 20 minutes, so we can write our equations in terms of time *T* by using the proportion

$$\frac{t}{2\pi} = \frac{T}{20}$$

Making Our Models More Realistic

Solving for *t* we have

$$t = \frac{\pi}{10}T$$

Substituting this expression for *t* into our parametric equations we have

$$x = 200 + 125\cos\left(\frac{\pi}{10}T - \frac{\pi}{2}\right)$$

$$y = 139 + 125\sin\left(\frac{\pi}{10}T - \frac{\pi}{2}\right)$$

These last equations give us a very accurate model of the path taken by someone riding on this Ferris wheel.

Making Our Models More Realistic

To graph these equations on our graphing calculator, we can use the following window:

$$\begin{aligned} \text{Radian mode:} \quad 0 \le T \le 20, \quad &\text{step} = 1 \\ -50 \le x \le 330, \ &\text{scale} = 20 \\ -50 \le y \le 330, \ &\text{scale} = 20 \end{aligned}$$

Using the zoom-square command, we have the graph shown in Figure 14.

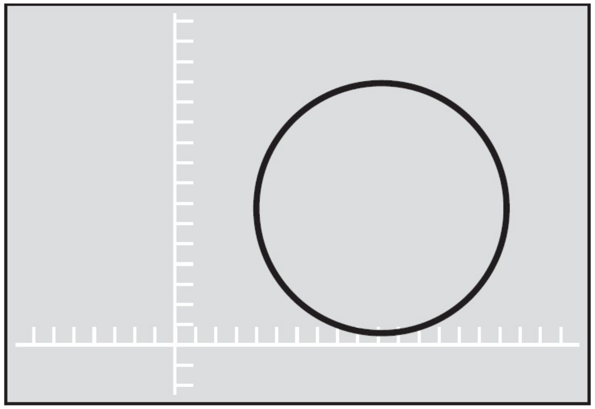


Figure 14

Making Our Models More Realistic

Figure 15 shows a trace of the graph. Tracing around this graph gives us the position of the rider at each minute of the 20-minute ride.

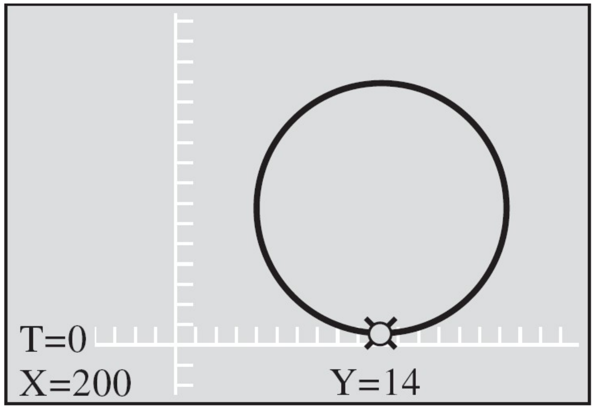


Figure 15

Parametric Equations and the Human Cannonball

Parametric Equations and the Human Cannonball

The human cannonball when shot from a cannon with an initial velocity of 50 miles per hour at an angle of 60° from the horizontal, will have a horizontal velocity of 25 miles per hour and an initial vertical velocity of 43 miles per hour (Figure 16).

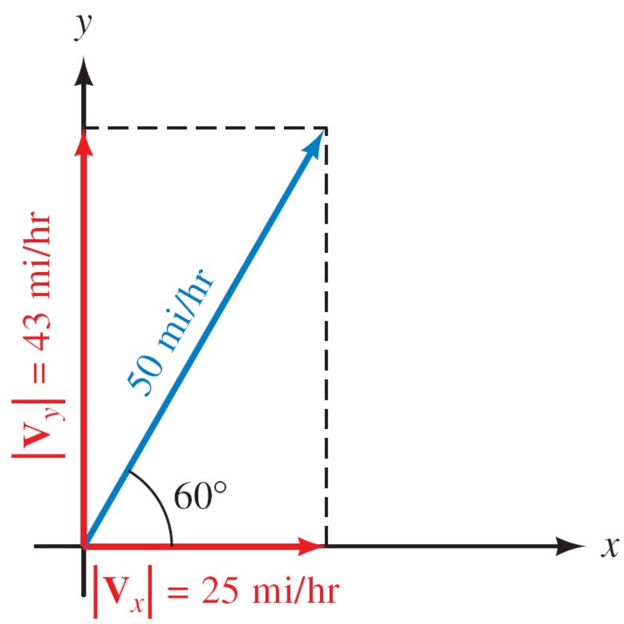


Figure 16

Parametric Equations and the Human Cannonball

We can generalize this so that a human cannonball, if shot from a cannon at $|\mathbf{V}_0|$ miles per hour at an angle of elevation θ, will have a horizontal speed of $|\mathbf{V}_0| \cos \theta$ miles per hour and an initial vertical speed of $|\mathbf{V}_0| \sin \theta$ miles per hour (Figure 17).

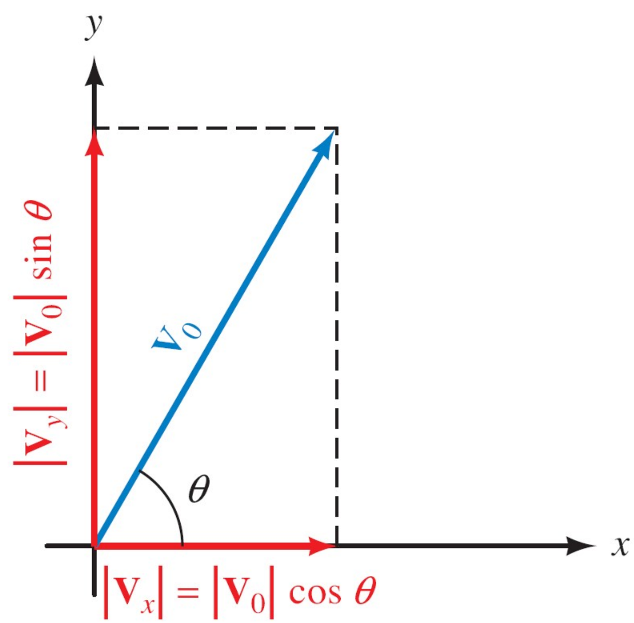


Figure 17

Parametric Equations and the Human Cannonball

Neglecting the resistance of air, the only force acting on the human cannonball is the force of gravity, which is an acceleration of 32 feet per second squared toward the earth.

Because the cannonball's horizontal speed is constant, we can find the distance traveled after *t* seconds by simply multiplying speed and time.

Therefore, the distance traveled horizontally after *t* seconds is

$$x = (|\mathbf{V}_0| \cos \theta)t$$

Parametric Equations and the Human Cannonball

To find the cannonball's vertical distance from the cannon after *t* seconds, we use a formula from physics:

$$y = (|\mathbf{V}_0| \sin \theta)t - \frac{1}{2}gt^2$$

where $g = 32$ ft/sec^2 (the acceleration of gravity on Earth)

This gives us the following set of parametric equations:

$$x = (|\mathbf{V}_0| \cos \theta)t$$
$$y = (|\mathbf{V}_0| \sin \theta)t - 16t^2$$

The equations describe the path of a human cannonball shot from a cannon at a speed of $|\mathbf{V}_0|$, at an angle of θ degrees from horizontal. So that the units will agree, $|\mathbf{V}_0|$ must be in feet per second because *t* is in seconds.