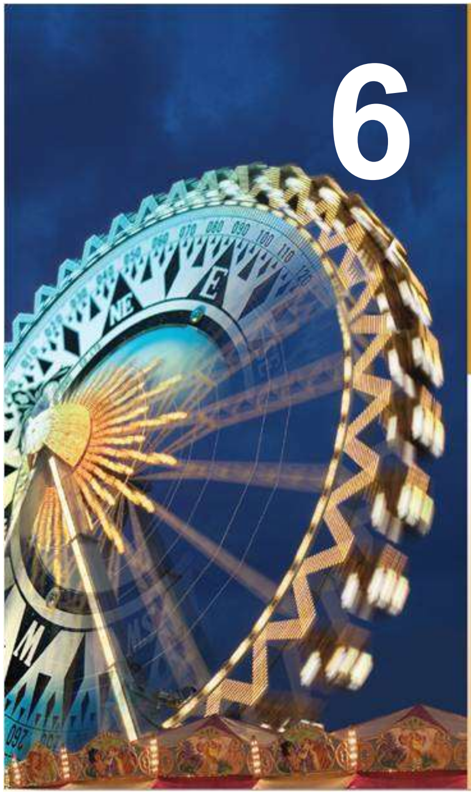

6 Equations

SECTION 6.3

Trigonometric Equations Involving Multiple Angles

Learning Objectives

1. Solve a simple trigonometric equation involving a multiple angle.

2. Use an identity to solve a trigonometric equation involving a multiple angle.

3. Solve a trigonometric equation involving a multiple angle by factoring or the quadratic formula.

4. Solve a real-life problem using a trigonometric equation.

Example 1

Solve $\cos 2\theta = \dfrac{\sqrt{3}}{2}$ $0° \leq \theta < 360°$.

Solution:

The equation cannot be simplified further. The reference angle is given by $\cos^{-1}(\sqrt{3}/2) = 30°$.

Because the cosine function is positive in QI or QIV, the expression 2θ must be coterminal with 30° or 330°.

Therefore, for any integer *k*, all solutions will have the form

$$2\theta = 30° + 360°k \qquad \text{or} \qquad 2\theta = 330° + 360°k$$

Example 1 – *Solution*

cont'd

$$\theta = \frac{30^\circ}{2} + \frac{360^\circ k}{2} \qquad \theta = \frac{330^\circ}{2} + \frac{360^\circ k}{2}$$

$$\theta = 15^\circ + 180^\circ k \qquad \theta = 165^\circ + 180^\circ k$$

Because we were asked only for values of θ between 0° and 360°, we substitute the appropriate integers in place of *k*.

$$\begin{array}{lll} \text{If} & k = 0 \\ \text{then} & \theta = 15^\circ + 180^\circ(0) = 15^\circ \\ \text{or} & \theta = 165^\circ + 180^\circ(0) = 165^\circ \end{array} \qquad \begin{array}{ll} \text{If} & k = 1 \\ \text{then} & \theta = 15^\circ + 180^\circ(1) = 195^\circ \\ \text{or} & \theta = 165^\circ + 180^\circ(1) = 345^\circ \end{array}$$

For all other values of *k*, θ will fall outside the given interval.

Example 3

Solve $\sin 2x \cos x + \cos 2x \sin x = \frac{\sqrt{2}}{2}$, if $0 \leq x < 2\pi$.

Solution:

We can simplify the left side by using the formula for $\sin (A + B)$.

$$\sin 2x \cos x + \cos 2x \sin x = \frac{\sqrt{2}}{2}$$

$$\sin (2x + x) = \frac{\sqrt{2}}{2}$$

$$\sin 3x = \frac{\sqrt{2}}{2}$$

Example 3 – *Solution*

cont'd

First we find *all* possible solutions for *x*:

$$3x = \frac{\pi}{4} + 2k\pi \qquad \text{or} \qquad 3x = \frac{3\pi}{4} + 2k\pi \qquad k \text{ is any integer}$$

$$x = \frac{\pi}{12} + \frac{2k\pi}{3} \qquad \text{or} \qquad x = \frac{\pi}{4} + \frac{2k\pi}{3} \qquad \text{Divide by 3}$$

Example 3 – *Solution*

cont'd

To find those solutions that lie in the interval $0 \le x < 2\pi$, we let k take on values of 0, 1, and 2. Doing so results in the following solutions (Figure 5):

$$x = \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}, \frac{17\pi}{12}, \text{ and } \frac{19\pi}{12}$$

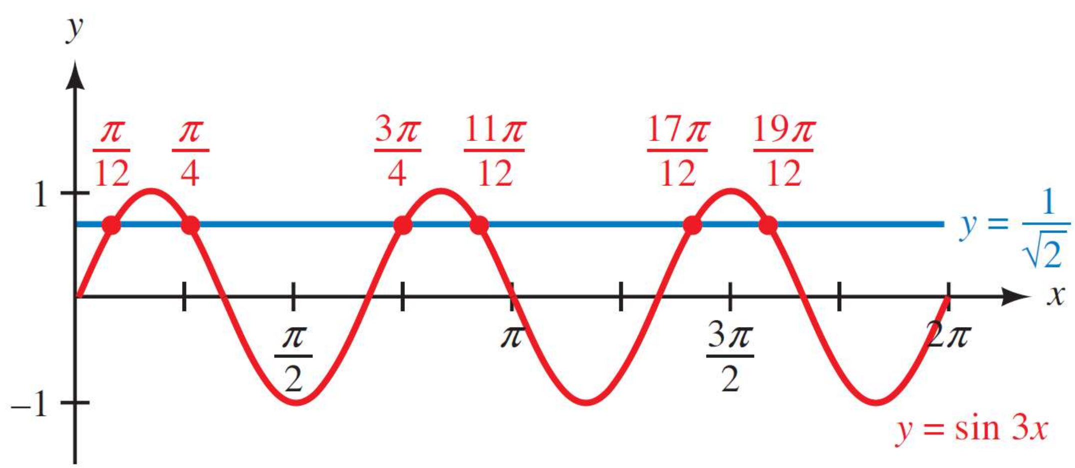

Figure 5

Example 4

Find all solutions to $2\sin^2 3\theta - \sin 3\theta - 1 = 0$, if θ is measured in degrees.

Solution:

We have an equation that is quadratic in $\sin 3\theta$. We factor and solve as usual.

$$2\sin^2 3\theta - \sin 3\theta - 1 = 0 \qquad \text{Standard form}$$

$$(2\sin 3\theta + 1)(\sin 3\theta - 1) = 0 \qquad \text{Factor}$$

$$2\sin 3\theta + 1 = 0 \quad \text{or} \quad \sin 3\theta - 1 = 0 \qquad \text{Set factors to 0}$$

Example 4 – *Solution*

cont'd

$$\sin 3\theta = -\frac{1}{2} \qquad \text{or} \qquad \sin 3\theta = 1$$

$3\theta = 210° + 360°k$ or $3\theta = 330° + 360°k$ or $3\theta = 90° + 360°k$

$\theta = 70° + 120°k$ or $\theta = 110° + 120°k$ or $\theta = 30° + 120°k$

Example 7

Table 3 shows the average monthly attendance at Lake Nacimiento in California.

Month	Attendance
January	6,500
February	6,600
March	15,800
April	26,000
May	38,000
June	36,000
July	31,300
August	23,500
September	12,000
October	4,000
November	900
December	2,100

Table 3

Example 7

cont'd

The average monthly attendance at Lake Nacimiento could be modeled by the function

$$y = 19{,}450 + 18{,}550 \cos\left(\frac{\pi}{6}x - \frac{5\pi}{6}\right), \qquad 0 \le x \le 12$$

where x is the month, with $x = 1$ corresponding to January.

Use the model to determine the percentage of the year that the average attendance is at least 25,000.

Example 7 – *Solution*

To begin, we will find the values of *x* for which *y* is exactly 25,000. To do so, we must solve the equation

$$19{,}450 + 18{,}550 \cos\left(\frac{\pi}{6}x - \frac{5\pi}{6}\right) = 25{,}000$$

First we isolate the trigonometric function.

$$19{,}450 + 18{,}550 \cos\left(\frac{\pi}{6}x - \frac{5\pi}{6}\right) = 25{,}000$$

$$18{,}550 \cos\left(\frac{\pi}{6}x - \frac{5\pi}{6}\right) = 5{,}550$$

$$\cos\left(\frac{\pi}{6}x - \frac{5\pi}{6}\right) = 0.2992$$

Example 7 – *Solution*

cont'd

The reference angle is $\cos^{-1}(0.2992) \approx 1.27$ radians. The cosine is positive in QI and QIV, so we have

$$\frac{\pi}{6}x - \frac{5\pi}{6} = 1.27 \qquad \text{or} \qquad \frac{\pi}{6}x - \frac{5\pi}{6} = 2\pi - 1.27$$

$$\frac{\pi}{6}x = 3.89 \qquad\qquad \frac{\pi}{6}x - \frac{5\pi}{6} = 5.01$$

$$x = 7.43 \qquad\qquad \frac{\pi}{6}x = 7.63$$

$$x = 14.57$$

The solution $x = 14.57$ is outside the given interval $0 \leq x \leq 12$. However, because the period is 12, adding or subtracting any multiple of 12 to 14.57 will give another solution.

Example 7 – *Solution*

cont'd

Thus, $x = 14.57 - 12 = 2.57$ is a valid solution within the given interval.

Looking at Figure 7, we see that y will be at least 25,000 between $x = 2.57$ and $x = 7.43$, inclusive.

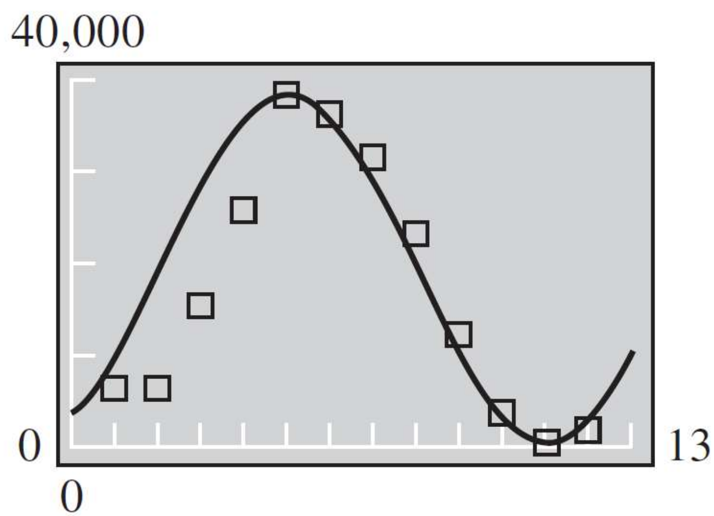

Figure 7

This gives us

$$7.43 - 2.57 = 4.86 \text{ months}$$

or

$$\frac{4.86}{12} = 0.405 = 40.5\% \text{ of the year}$$