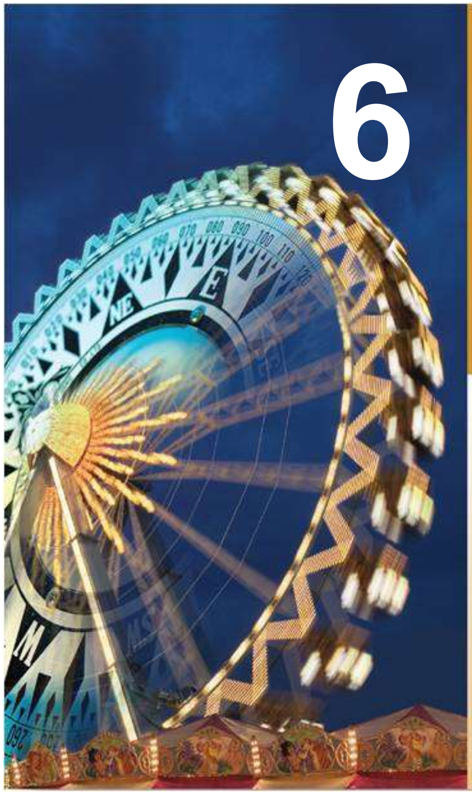

Equations

SECTION 6.1

Solving Trigonometric Equations

Learning Objectives

1. Solve a simple trigonometric equation.
2. Use factoring to solve a trigonometric equation.
3. Use the quadratic formula to solve a trigonometric equation.
4. Solve a trigonometric equation involving a horizontal translation.

Solving Trigonometric Equations

The solution set for an equation is the set of all numbers that, when used in place of the variable, make the equation a true statement.

Most equations, however, are not identities. They are only true for certain values of the variable. We call these types of equations *conditional equations*.

Solving these equations was accomplished by applying two important properties: the *addition property of equality* and the *multiplication property of equality*.

Solving Trigonometric Equations

These two properties were stated as follows:

ADDITION PROPERTY OF EQUALITY

For any three algebraic expressions A, B, and C

$$\begin{array}{ll} \text{If} & A = B \\ \text{then} & A + C = B + C \end{array}$$

In words: Adding the same quantity to both sides of an equation will not change the solution set.

MULTIPLICATION PROPERTY OF EQUALITY

For any three algebraic expressions A, B, and C, with $C \neq 0$,

$$\begin{array}{ll} \text{If} & A = B \\ \text{then} & AC = BC \end{array}$$

In words: Multiplying both sides of an equation by the same nonzero quantity will not change the solution set.

Solving Trigonometric Equations

The process of solving trigonometric equations is very similar to the process of solving algebraic equations.

With trigonometric equations, we look for values of an *angle* that will make the equation into a true statement.

We usually begin by solving for a specific trigonometric function of that angle and then use the concepts we have developed earlier to find the angle.

Here are some examples that illustrate this procedure.

Example 2

Solve $2 \sin x - 1 = 0$ for *x*.

Solution:

We can solve for sin *x* using our methods from algebra. We then use our knowledge of trigonometry to find *x* itself.

$$2 \sin x - 1 = 0$$

$$2 \sin x = 1$$

$$\sin x = \frac{1}{2}$$

Example 2 – *Solution*

cont'd

From Figure 1 we can see that if we are looking for radian solutions between 0 and 2π, then x is either $\pi/6$ or $5\pi/6$.

On the other hand, if we want degree solutions between 0° and 360°, then our solutions will be 30° and 150°.

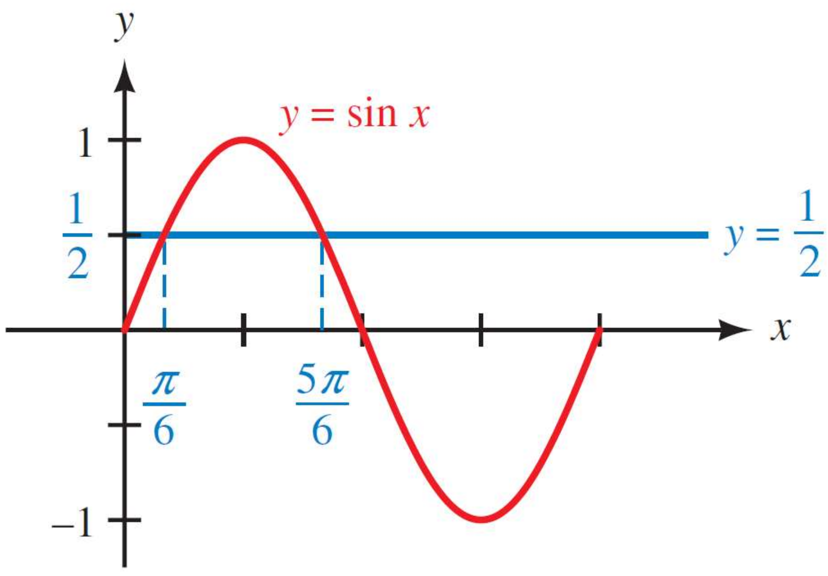

Figure 1

Without the aid of Figure 1, we would reason that, because $\sin x = \frac{1}{2}$, the reference angle for x is 30°.

Example 2 – *Solution*

cont'd

Then, because $\frac{1}{2}$ is a positive number and the sine function is positive in QI and QII, x must be 30° or 150° (Figure 2).

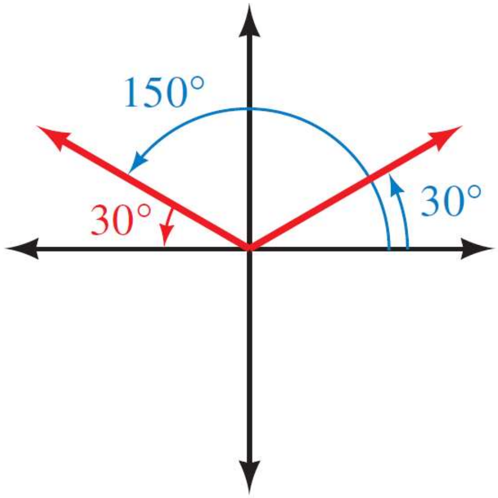

Figure 2

Solutions Between 0° and 360° or 0 and 2π

In Degrees	In Radians
$x = 30°$ or $x = 150°$	$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

Example 2 – *Solution*

cont'd

Because the sine function is periodic with period 2π (or 360°), any angle coterminal with $x = \pi/6$ (or 30°) or $x = 5\pi/6$ (or 150°) will also be a solution of the equation.

For any integer k, adding $2\pi k$ (or $360°k$) will result in a coterminal angle. Therefore, we can represent *all* solutions to the equation as follows.

All Solutions (k Is an Integer)

	In Degrees		In Radians
	$x = 30° + 360°k$		$x = \dfrac{\pi}{6} + 2k\pi$
or	$x = 150° + 360°k$	or	$x = \dfrac{5\pi}{6} + 2k\pi$

Example 4

Find all degree solutions to $\cos(A - 25^\circ) = -\dfrac{\sqrt{2}}{2}$.

Solution:

The reference angle is given by $\cos^{-1}(\sqrt{2}/2) = 45^\circ$.

Because the cosine function is negative in QII or QIII, the expression *A* – 25° must be coterminal with 135° or 225°.

Therefore,

$$A - 25^\circ = 135^\circ + 360^\circ k \qquad \text{or} \qquad A - 25^\circ = 225^\circ + 360^\circ k$$

for any integer *k*.

Example 4 – *Solution*

cont'd

We can now solve for A by adding 25° to both sides.

$$A - 25° = 135° + 360°k \qquad \text{or} \qquad A - 25° = 225° + 360°k$$

$$A = 160° + 360°k \qquad\qquad A = 250° + 360°k$$

Solving Trigonometric Equations

The next kind of trigonometric equation we will solve is quadratic in form.

In algebra, the two most common methods of solving quadratic equations are factoring and applying the quadratic formula.

Here is an example that reviews the factoring method.

Example 7

Solve $2\cos^2 t - 9\cos t = 5$, if $0 \le t < 2\pi$.

Solution:

The fact that $0 \le t \le 2\pi$ indicates we are to write our solutions in radians.

$$2\cos^2 t - 9\cos t = 5$$

$$2\cos^2 t - 9\cos t - 5 = 0 \qquad \text{Standard form}$$

$$(2\cos t + 1)(\cos t - 5) = 0 \qquad \text{Factor}$$

Example 7 – *Solution*

cont'd

$$2\cos t + 1 = 0 \qquad \text{or} \qquad \cos t - 5 = 0 \qquad \text{Set each factor to 0}$$

$$\cos t = -\frac{1}{2} \qquad \text{or} \qquad \cos t = 5 \qquad \text{Isolate } \cos t$$

The first result, $\cos t = -\frac{1}{2}$, gives us a reference angle of

$$\hat{\theta} = \cos^{-1}(1/2) = \pi/3$$

Example 7 – *Solution*

cont'd

Because cos t is negative, t must terminate in QII or QIII (Figure 7). Therefore,

$$t = \pi - \pi/3 = 2\pi/3 \qquad \text{or}$$

$$t = \pi + \pi/3 = 4\pi/3$$

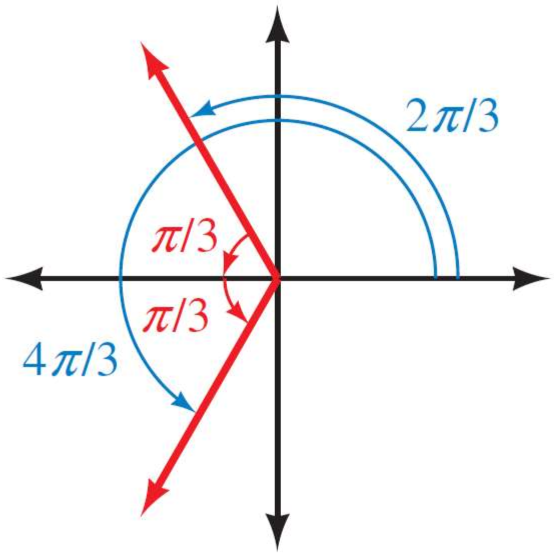

Figure 7

The second result, cos t = 5, has no solution.

For any value of t, cos t must be between –1 and 1. It can never be 5.

Example 8

Solve $2\sin^2\theta + 2\sin\theta - 1 = 0$, if $0 \le \theta < 2\pi$.

Solution:

The equation is already in standard form. If we try to factor the left side, however, we find it does not factor. We must use the quadratic formula.

The quadratic formula states that the solutions to the equation

$$ax^2 + bx + c = 0$$

will be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 8 – *Solution*

cont'd

In our case, the coefficients *a*, *b*, and *c* are

$$a = 2, \qquad b = 2, \qquad c = -1$$

Using these numbers, we can solve for sin θ as follows:

$$\sin\theta = \frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-2 \pm \sqrt{12}}{4}$$

$$= \frac{-2 \pm 2\sqrt{3}}{4} \qquad = \frac{-1 \pm \sqrt{3}}{2}$$

Example 8 – *Solution*

cont'd

Using the approximation $\sqrt{3} = 1.7321,$ we arrive at the following decimal approximations for sin θ:

$$\sin\theta = \frac{-1 + 1.7321}{2} \quad \text{or} \quad \sin\theta = \frac{-1 - 1.7321}{2}$$

$$\sin\theta = 0.3661 \quad \text{or} \quad \sin\theta = -1.3661$$

We will not obtain any solutions from the second expression, sin θ = –1.3661, because sin θ must be between –1 and 1. For sin θ = 0.3661, we use a calculator to find the angle whose sine is nearest to 0.3661.

Example 8 – *Solution*

cont'd

That angle is approximately 0.37 radian, and it is the reference angle for θ.

Since sin θ is positive, θ must terminate in QI or QII (Figure 8).

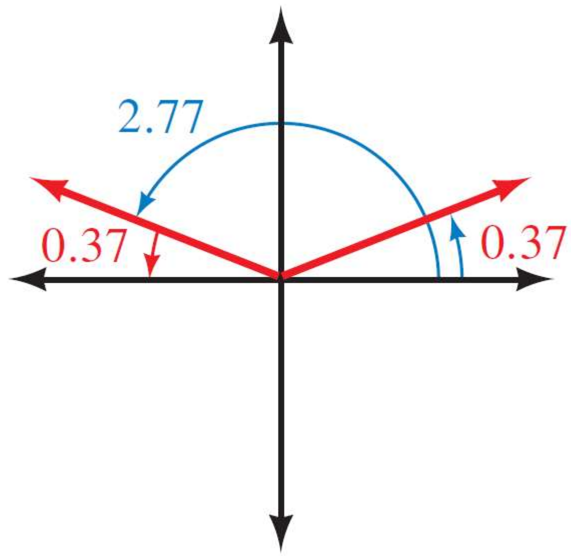

Figure 8

Therefore,

$$\theta = 0.37 \qquad \text{or} \qquad \theta = \pi - 0.37 = 2.77$$