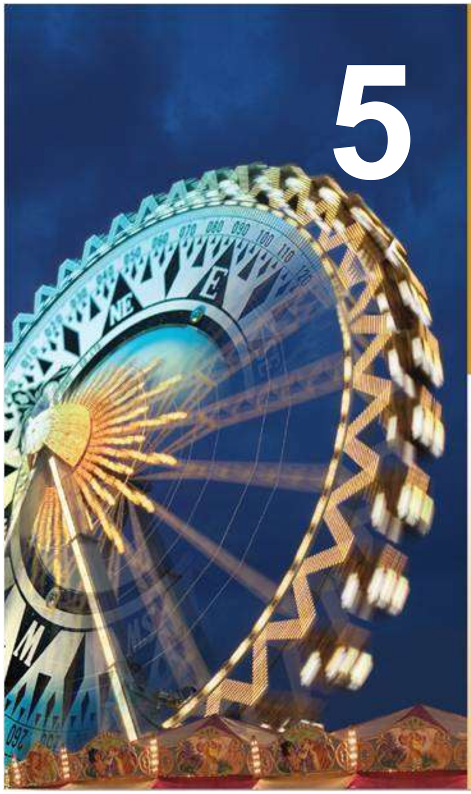

5 Identities and Formulas

SECTION 5.4

Half-Angle Formulas

Learning Objectives

1. Determine the quadrant in which a half-angle terminates.
2. Use a half-angle formula to find the exact value of a trigonometric function.
3. Use a half-angle formula to graph an equation.
4. Prove an equation is an identity using a half-angle formula.

Half-Angle Formulas

In this section, we will derive formulas for sin $\frac{A}{2}$ and cos $\frac{A}{2}$.

These formulas are called *half-angle* formulas and are derived from the double-angle formulas for cos 2*A*.

Because every value of *x* can be written as $\frac{1}{2}$ of some other number *A*, we can replace *x* with *A*/2. This is equivalent to saying 2*x* = *A*.

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}$$

This last expression is the half-angle formula for sin $\frac{A}{2}$.

Half-Angle Formulas

To find the half-angle formula for cos $\frac{A}{2}$, we solve $\cos 2x = 2\cos^2 x - 1$ for cos x and then replace x with $A/2$ (and $2x$ with A). Without showing the steps involved in this process, here is the result:

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

In both half-angle formulas, the sign in front of the radical, + or –, is determined by the quadrant in which $A/2$ terminates.

Example 1

If cos $A = \frac{3}{5}$ with 270° < A < 360°, find sin $\frac{A}{2}$, cos $\frac{A}{2}$, and tan $\frac{A}{2}$.

Solution:

First of all, we determine the quadrant in which $A/2$ terminates.

$$\text{If} \qquad 270° < A < 360°$$

$$\text{then} \qquad \frac{270°}{2} < \frac{A}{2} < \frac{360°}{2}$$

$$135° < \frac{A}{2} < 180° \qquad \text{so } \frac{A}{2} \in \text{QII}$$

Example 1 – *Solution*

cont'd

In QII, sine is positive, and cosine is negative. Using the half-angle formulas, we have

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-3/5}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}} = -\sqrt{\frac{1+3/5}{2}} = -\sqrt{\frac{4}{5}} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

Example 1 – *Solution*

cont'd

Now we can use a ratio identity to find $\tan \frac{A}{2}$.

$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}$$

$$= \frac{\frac{\sqrt{5}}{5}}{-\frac{2\sqrt{5}}{5}}$$

$$= -\frac{1}{2}$$

Half-Angle Formulas

If we let $\theta = \frac{A}{2}$ in this identity, we obtain a formula for tan $\frac{A}{2}$ that involves only sin *A* and cos *A*. Here it is.

$$\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}$$

If we multiply the numerator and denominator of the right side of this formula by 1 + cos *A* and simplify the result, we have a second formula for tan $\frac{A}{2}$.

$$\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}$$

Example 4

Graph $y = 4\cos^2 \frac{x}{2}$ from $x = 0$ to $x = 4\pi$.

Solution:

Applying our half-angle formula for $\cos \frac{x}{2}$ to the right side, we have

$$y = 4\cos^2 \frac{x}{2} = 4\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2$$

$$= 4\left(\frac{1+\cos x}{2}\right)$$

$$= 2 + 2\cos x$$

Example 4 – *Solution*

cont'd

The graph of $y = 2 + 2 \cos x$ has an amplitude of 2 and a vertical translation 2 units upward.

The graph is shown in Figure 1.

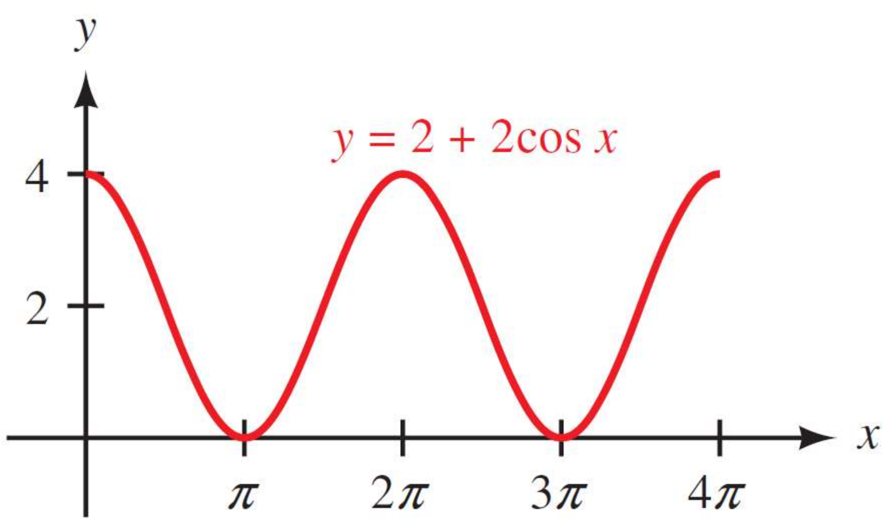

Figure 1

Example 5

Prove $\sin^2 \frac{x}{2} = \frac{\tan x - \sin x}{2 \tan x}$.

Solution:

We can use a half-angle formula on the left side. In this case, because we have $\sin^2(x/2)$, we write the half-angle formula without the square root sign.

After that, we multiply the numerator and denominator on the left side by tan *x* because the right side has tan *x* in both the numerator and the denominator.

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2} \qquad \text{Square of half-angle formula}$$

Example 5 – *Solution*

cont'd

$$= \frac{\tan x}{\tan x} \cdot \frac{1 - \cos x}{2}$$ Multiply numerator and denominator by $\tan x$

$$= \frac{\tan x - \tan x \cos x}{2 \tan x}$$ Distributive property

$$= \frac{\tan x - \sin x}{2 \tan x}$$ $\tan x \cos x$ is $\sin x$