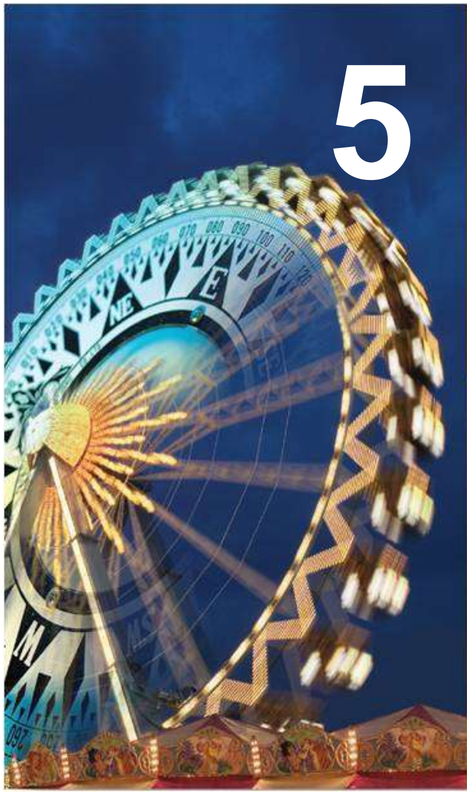

5 Identities and Formulas

SECTION 5.3

Double-Angle Formulas

Learning Objectives

1. Use a double-angle formula to find the exact value of a trigonometric function.
2. Simplify an expression using a double-angle formula.
3. Use a double-angle formula to graph an equation.
4. Prove an equation is an identity using a double-angle formula.

Double-Angle Formulas

We will begin this section by deriving the formulas for sin 2*A* and cos 2*A* using the formulas for sin (*A* + *B*) and cos (*A* + *B*).

The formulas we derive for sin 2*A* and cos 2*A* are called *double-angle* formulas. Here is the derivation of the formula for sin 2*A*.

$$\begin{aligned} \sin 2A &= \sin(A + A) && \text{Write } 2A \text{ as } A + A \\ &= \sin A \cos A + \cos A \sin A && \text{Sum formula} \\ &= \sin A \cos A + \sin A \cos A && \text{Commutative property} \\ &= 2 \sin A \cos A \end{aligned}$$

Double-Angle Formulas

The last line gives us our first double-angle formula.

$$\boxed{\sin 2A = 2 \sin A \cos A}$$

The first thing to notice about this formula is that it indicates the 2 in sin 2*A* *cannot* be factored out and written as a coefficient. That is,

$$\sin 2A \neq 2 \sin A$$

Example 1

If $\sin A = \frac{3}{5}$ with *A* in QII, find sin 2*A*.

Solution:

To apply the formula for sin 2*A*, we must first find cos *A*. Because *A* terminates in QII, cos *A* is negative.

$$\cos A = \pm\sqrt{1 - \sin^2 A}$$

$$= -\sqrt{1 - \left(\frac{3}{5}\right)^2} \qquad A \in \text{QII, so } \cos A \text{ is negative}$$

$$= -\frac{4}{5}$$

Example 1 – *Solution*

cont'd

Now we can apply the formula for sin 2*A*.

$$\sin 2A = 2 \sin A \cos A$$

$$= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$$

$$= -\frac{24}{25}$$

Example 3

Prove $\sin 2x = \dfrac{2 \cot x}{1 + \cot^2 x}$.

Solution:

$$\frac{2 \cot x}{1 + \cot^2 x} = \frac{2 \cdot \dfrac{\cos x}{\sin x}}{1 + \dfrac{\cos^2 x}{\sin^2 x}}$$ Ratio identity

$$= \frac{2 \sin x \cos x}{\sin^2 x + \cos^2 x}$$ Multiply numerator and denominator by $\sin^2 x$

$$= 2 \sin x \cos x$$ Pythagorean identity

$$= \sin 2x$$ Double-angle identity

Double-Angle Formulas

There are three forms of the double-angle formula for $\cos 2A$. The first involves both $\sin A$ and $\cos A$, the second involves only $\cos A$, and the third involves only $\sin A$.

Here is how we obtain the three formulas.

$$\begin{aligned}\cos 2A &= \cos(A + A) && \text{Write } 2A \text{ as } A + A \\ &= \cos A \cos A - \sin A \sin A && \text{Sum formula} \\ &= \cos^2 A - \sin^2 A\end{aligned}$$

Here are the three forms of the double-angle formula for $\cos 2A$.

$$\boxed{\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A && \text{First form} \\ &= 2\cos^2 A - 1 && \text{Second form} \\ &= 1 - 2\sin^2 A && \text{Third form}\end{aligned}}$$

Example 6

Graph $y = 3 - 6 \sin^2 x$ from $x = 0$ to $x = 2\pi$.

Solution:

To write the equation in the form $y = A \cos Bx$, we factor 3 from each term on the right side and then apply the formula for $\cos 2A$ to the remaining expression to write it as a single trigonometric function.

$$
\begin{aligned}
y &= 3 - 6 \sin^2 x \\
&= 3(1 - 2 \sin^2 x) && \text{Factor 3 from each term} \\
&= 3 \cos 2x && \text{Double-angle formula}
\end{aligned}
$$

Example 6 – *Solution*

cont'd

The graph of $y = 3 \cos 2x$ will have an amplitude of 3 and a period of $2\pi/2 = \pi$.

The graph is shown in Figure 1.

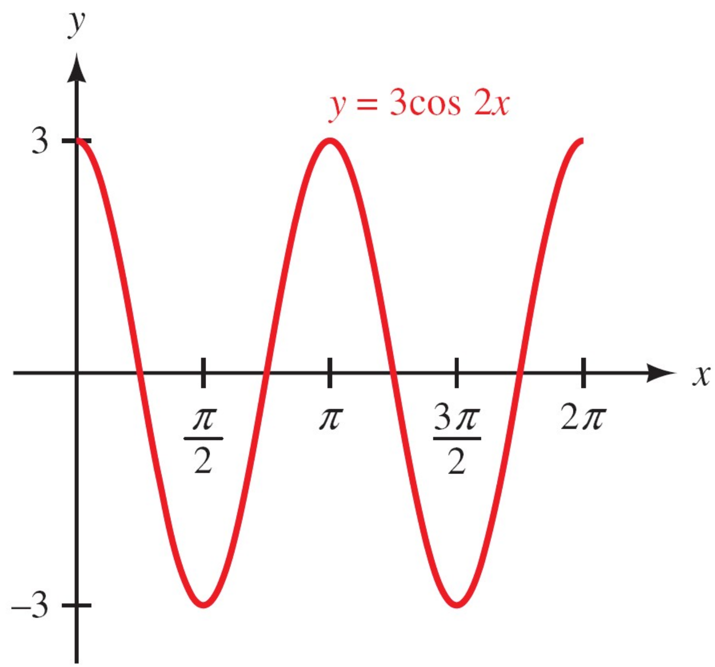

Figure 1

Double-Angle Formulas

Our double-angle formula for tan $2A$ is

$$\boxed{\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}}$$

Example 8

Simplify $\dfrac{2 \tan 15^\circ}{1 - \tan^2 15^\circ}$.

Solution:

The expression has the same form as the right side of our double-angle formula for tan 2*A*. Therefore,

$$\frac{2 \tan 15^\circ}{1 - \tan^2 15^\circ} = \tan (2 \cdot 15^\circ)$$

$$= \tan 30^\circ$$

$$= \frac{\sqrt{3}}{3}$$