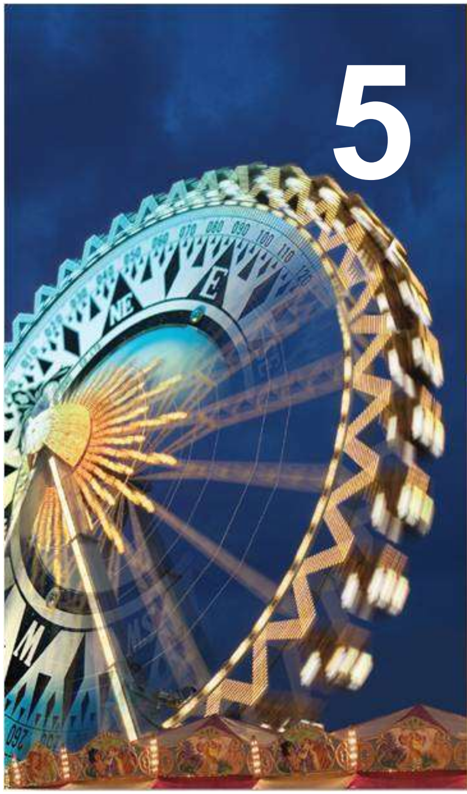

5 Identities and Formulas

SECTION 5.2

Sum and Difference Formulas

Learning Objectives

1. Use a sum or difference formula to find the exact value of a trigonometric function.
2. Prove an equation is an identity using a sum or difference formula.
3. Simplify an expression using a sum or difference formula.
4. Use a sum or difference formula to graph an equation.

Sum and Difference Formulas

We begin by drawing angle *A* in standard position and then adding *B* to it. Next we draw –*B* in standard position.

Figure 1 shows these angles in relation to the unit circle.

Note that the points on the unit circle through which the terminal sides of the angles *A*, *A* + *B*, and –*B* pass have been labeled with the sines and cosines of those angles.

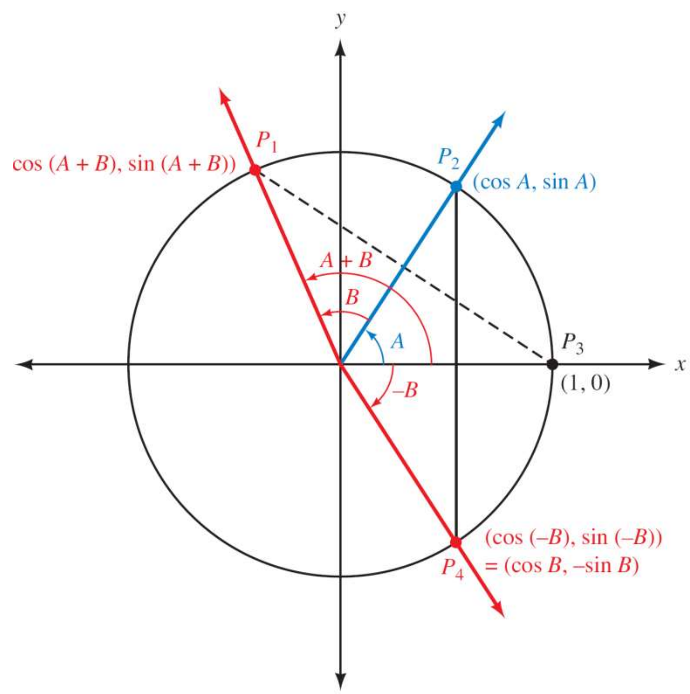

Figure 1

Sum and Difference Formulas

To derive the formula for cos (A + B), we simply have to see that length P_1P_3 is equal to length P_2P_4. (From geometry, they are chords cut off by equal central angles.)

$$P_1 P_3 = P_2 P_4$$

Squaring both sides gives us

$$(P_1 P_3)^2 = (P_2 P_4)^2$$

Now, applying the distance formula, we have

$$[\cos (A + B) - 1]^2 + [\sin (A + B) - 0]^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$

Let’s call this Equation 1.

Sum and Difference Formulas

Considering the Equation 1, expanding it, and then simplifying by using the first Pythagorean identity gives us

$$\boxed{\cos(A + B) = \cos A \cos B - \sin A \sin B}$$

This is the first formula in a series of formulas for trigonometric functions of the sum or difference of two angles.

Example 1

Find the exact value for cos 75°.

Solution:

We write 75° as 45° + 30° and then apply the formula for cos (A + B).

$$
\begin{aligned}
\cos 75^\circ &= \cos(45^\circ + 30^\circ) \\
&= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
&= \frac{\sqrt{6} - \sqrt{2}}{4}
\end{aligned}
$$

Example 3

Write $\cos 3x \cos 2x - \sin 3x \sin 2x$ as a single cosine.

Solution:

We apply the formula for cos (*A* + *B*) in the reverse direction from the way we applied it in the first example.

$$\begin{aligned}\cos 3x \cos 2x - \sin 3x \sin 2x &= \cos (3x + 2x) \\ &= \cos 5x\end{aligned}$$

Sum and Difference Formulas

Here is the derivation of the formula for cos (*A* – *B*). It involves the formula for cos (*A* + *B*) and the formulas for even and odd functions.

$$\cos(A - B) = \cos[A + (-B)] \qquad \text{Write } A - B \text{ as a sum}$$

$$= \cos A \cos(-B) - \sin A \sin(-B) \qquad \text{Sum formula}$$

$$= \cos A \cos B - \sin A(-\sin B) \qquad \text{Cosine is an even function, sine is odd}$$

$$= \cos A \cos B + \sin A \sin B$$

The only difference in the formulas for the expansion of cos (*A* + *B*) and cos (*A* – *B)* is the sign between the two terms.

Sum and Difference Formulas

Here are both formulas.

$$\boxed{\begin{aligned} \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \cos(A - B) &= \cos A \cos B + \sin A \sin B \end{aligned}}$$

Again, both formulas are important and should be memorized.

Example 4

Show that $\cos(90° - A) = \sin A$.

Solution:

We will need this formula when we derive the formula for sin (*A* + *B*).

$$\begin{aligned} \cos(90° - A) &= \cos 90° \cos A + \sin 90° \sin A \\ &= 0 \cdot \cos A + 1 \cdot \sin A \\ &= \sin A \end{aligned}$$

Sum and Difference Formulas

Note that the formula we just derived is not a new formula. The angles $90° – A$ and A are complementary angles, and we have known by the Cofunction Theorem that the sine of an angle sine of an angle is always equal to the cosine of its complement.

We could also state it this way:

$$\sin (90° - A) = \cos A$$

We can use this information to derive the formula for $\sin (A + B)$. To understand this derivation, you must recognize that $A + B$ and $90° – (A + B)$ are complementary angles.

Sum and Difference Formulas

$$\sin(A + B) = \cos[90° - (A + B)]$$ The sine of an angle is the cosine of its complement

$$= \cos[90° - A - B]$$ Remove parentheses

$$= \cos[(90° - A) - B]$$ Regroup within brackets

Now we expand using the formula for the cosine of a difference.

$$= \cos(90° - A)\cos B + \sin(90° - A)\sin B$$

$$= \sin A \cos B + \cos A \sin B$$

This gives us an expansion formula for sin (*A* + *B*).

$$\boxed{\sin(A + B) = \sin A \cos B + \cos A \sin B}$$

This is the formula for the sine of a sum.

Sum and Difference Formulas

To find the formula for sin (A – B), we write A – B as A + (–B) and proceed as follows:

$$\begin{aligned}\sin(A - B) &= \sin(A + (-B)) \\ &= \sin A \cos(-B) + \cos A \sin(-B) \\ &= \sin A \cos B - \cos A \sin B\end{aligned}$$

This gives us the formula for the sine of a difference.

$$\boxed{\sin(A - B) = \sin A \cos B - \cos A \sin B}$$

Example 5

Graph $y = 4 \sin 5x \cos 3x - 4 \cos 5x \sin 3x$ from $x = 0$ to $x = 2\pi$.

Solution:

To write the equation in the form $y = A \sin Bx$, we factor 4 from each term on the right and then apply the formula for sin (A – B) to the remaining expression to write it as a single trigonometric function.

$$
\begin{aligned}
y &= 4 \sin 5x \cos 3x - 4 \cos 5x \sin 3x \\
&= 4 (\sin 5x \cos 3x - \cos 5x \sin 3x) \\
&= 4 \sin (5x - 3x) \\
&= 4 \sin 2x
\end{aligned}
$$

Example 5 – *Solution*

cont'd

The graph of $y = 4 \sin 2x$ will have an amplitude of 4 and a period of $2\pi/2 = \pi$.

The graph is shown in Figure 2.

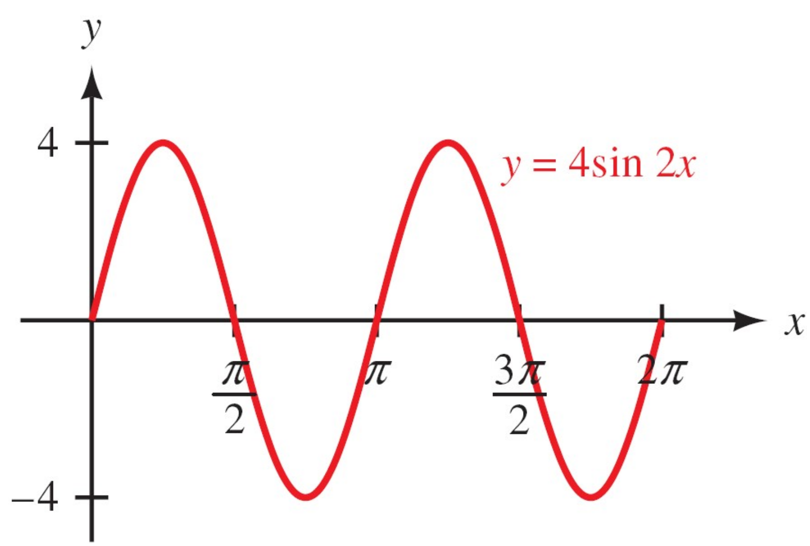

Figure 2

Example 7

If $\sin A = \frac{3}{5}$ with *A* in QI and $\cos B = -\frac{5}{13}$ with *B* in QIII, find sin (*A* + *B*), cos (*A* + *B*), and tan (*A* + *B*).

Solution:

We have sin *A* and cos *B*. We need to find cos *A* and sin *B* before we can apply any of our formulas. Some equivalent forms of our first Pythagorean identity will help here.

If $\sin A = \frac{3}{5}$ with *A* in QI, then

$$\cos A = \pm\sqrt{1 - \sin^2 A}$$

Example 7 – *Solution*

cont'd

$$= +\sqrt{1 - \left(\frac{3}{5}\right)^2} \qquad A \in \text{QI, so } \cos A \text{ is positive}$$

$$= \frac{4}{5}$$

If $\cos B = -\frac{5}{13}$ with *B* in QIII, then

$$\sin B = \pm\sqrt{1 - \cos^2 B}$$

$$= -\sqrt{1 - \left(-\frac{5}{13}\right)^2} \qquad B \in \text{QIII, so } \sin B \text{ is negative}$$

$$= -\frac{12}{13}$$

Example 7 – *Solution*

cont'd

We have

$$\sin A = \frac{3}{5} \qquad \sin B = -\frac{12}{13}$$

$$\cos A = \frac{4}{5} \qquad \cos B = -\frac{5}{13}$$

Therefore,

$$\begin{aligned}\sin(A + B) &= \sin A \cos B + \cos A \sin B \\ &= \frac{3}{5}\left(-\frac{5}{13}\right) + \frac{4}{5}\left(-\frac{12}{13}\right) \\ &= -\frac{63}{65}\end{aligned}$$

Example 7 – *Solution*

cont'd

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{4}{5}\left(-\frac{5}{13}\right) - \frac{3}{5}\left(-\frac{12}{13}\right)$$

$$= \frac{16}{65}$$

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= \frac{-63/65}{16/65}$$

$$= -\frac{63}{16}$$

Example 7 – *Solution*

cont'd

Notice also that $A + B$ must terminate in QIV because

$$\sin(A + B) < 0 \qquad \text{and} \qquad \cos(A + B) > 0$$

Sum and Difference Formulas

The formula for tan (A + B) is

$$\boxed{\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

Because tangent is an odd function, the formula for tan (A – B) will look like this:

$$\boxed{\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$