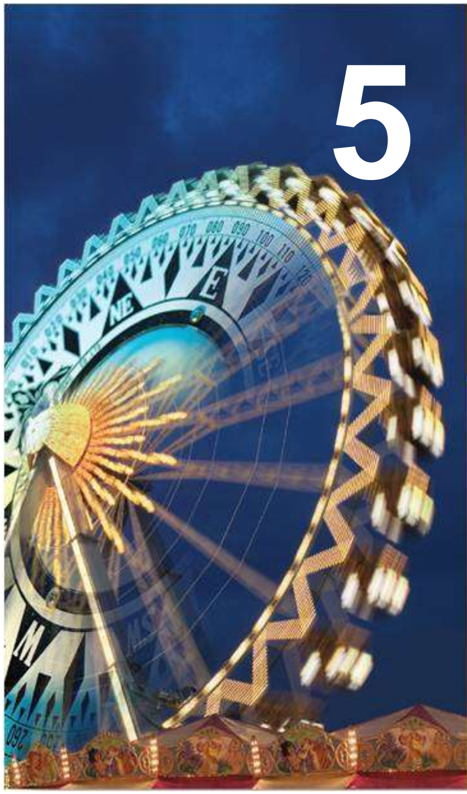

5 Identities and Formulas

SECTION 5.1

Proving Identities

Learning Objectives

1. Prove an equation is an identity.
2. Use a counterexample to prove an equation is not an identity.
3. Use a graphing calculator to determine if an equation appears to be an identity.

Proving Identities

Table 1 lists the basic identities and some of their more important equivalent forms.

	Basic Identities	Common Equivalent Forms
Reciprocal	$\csc\theta = \dfrac{1}{\sin\theta}$	$\sin\theta = \dfrac{1}{\csc\theta}$
	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$
	$\cot\theta = \dfrac{1}{\tan\theta}$	$\tan\theta = \dfrac{1}{\cot\theta}$
Ratio	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	
	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	
Pythagorean	$\cos^2\theta + \sin^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$ $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$ $\cos^2\theta = 1 - \sin^2\theta$ $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$
	$1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$	

Table 1

Example 2

Prove $\tan x + \cos x = \sin x\,(\sec x + \cot x)$.

Solution:

We can begin by applying the distributive property to the right side to multiply through by sin *x*.

Then we can change the right side to an equivalent expression involving only sin *x* and cos *x*.

$$\sin x\,(\sec x + \cot x) = \sin x \sec x + \sin x \cot x$$

$$= \sin x \cdot \frac{1}{\cos x} + \sin x \cdot \frac{\cos x}{\sin x} \qquad \text{Reciprocal and ratio identities}$$

Example 2 – *Solution*

cont'd

$$= \frac{\sin x}{\cos x} + \cos x \qquad \text{Multiply}$$

$$= \tan x + \cos x \qquad \text{Ratio identity}$$

In this case, we transformed the right side into the left side.

Proving Identities

GUIDELINES FOR PROVING IDENTITIES

1. It is usually best to work on the more complicated side first.
2. Look for trigonometric substitutions involving the basic identities that may help simplify things.
3. Look for algebraic operations, such as adding fractions, the distributive property, or factoring, that may simplify the side you are working with or that will at least lead to an expression that will be easier to simplify.
4. If you cannot think of anything else to do, change everything to sines and cosines and see if that helps.
5. Always keep an eye on the side you are not working with to be sure you are working toward it. There is a certain sense of direction that accompanies a successful proof.

Example 4

Prove $1 + \cos\theta = \dfrac{\sin^2\theta}{1 - \cos\theta}$.

Solution:

We begin this proof by applying an alternate form of the Pythagorean identity to the right side to write $\sin^2\theta$ as $1 - \cos^2\theta$.

Then we factor $1 - \cos^2\theta$ as the difference of two squares and reduce to lowest terms.

$$\frac{\sin^2\theta}{1 - \cos\theta} = \frac{1 - \cos^2\theta}{1 - \cos\theta} \qquad \text{Pythagorean identity}$$

Example 4 – *Solution*

cont'd

$$= \frac{(1 - \cos\theta)(1 + \cos\theta)}{1 - \cos\theta} \qquad \text{Factor}$$

$$= 1 + \cos\theta \qquad \text{Reduce}$$

Proving Identities

You can use your graphing calculator to decide if an equation is an identity or not. If the two expressions are indeed equal for all defined values of the variable, then they should produce identical graphs.

Although this does not constitute a proof, it does give strong evidence that the identity is true.

We can verify the identity in Example 4 by defining the expression on the left as a function Y_1 and the expression on the right as a second function Y_2.

Proving Identities

If your calculator is equipped with different graphing styles, set the style of Y_2 so that you will be able to distinguish the second graph from the first.

(In Figure 1, we have used the *path* style on a TI-84 for the second function.) Also, be sure your calculator is set to radian mode.

```
Plot1  Plot2  Plot3
 \Y1=1+cos(X)
-0Y2=(sin(X))^2/(1–cos(X))
 \Y3=
 \Y4=
 \Y5=
 \Y6=
 \Y7=
```

Figure 1

Proving Identities

Set your window variables so that

$$-2\pi \le x \le 2\pi, \text{ scale} = \pi/2; -4 \le y \le 4, \text{ scale} = 1$$

When you graph the functions, your calculator screen should look similar to Figure 2 (the small circle is a result of the path style in action). Observe that the two graphs are identical.

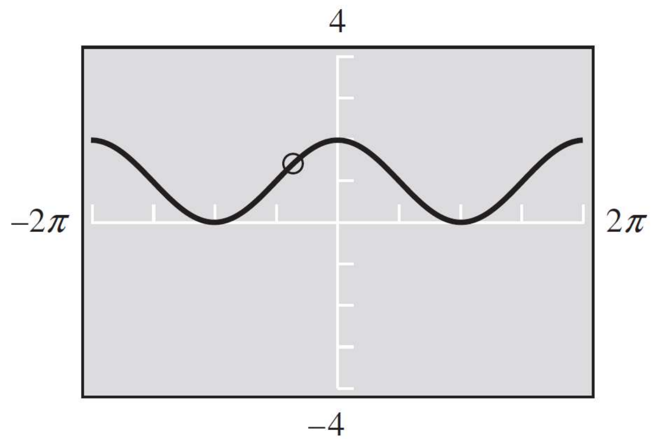

Figure 2

Proving Identities

If your calculator is not equipped with different graphing styles, it may be difficult to tell if the second graph really coincides with the first.

In this case you can trace the graph, and switch between the two functions at several points to convince yourself that the two graphs are indeed the same.

To show that a statement is *not an* identity is usually much simpler. All we must do is find a single value of the variable for which each expression is defined, but which makes the statement false. This is known as finding a *counterexample*.

Example 8

Show that $\cot^2\theta + \cos^2\theta = \cot^2\theta\cos^2\theta$ is not an identity by finding a counterexample.

Solution:

Because cot θ is undefined for $\theta = k\pi$, where k is any integer, we must choose some other value of θ as a counterexample.

Using $\theta = \pi/4$, we find

$$\cot^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} \stackrel{?}{=} \cot^2\frac{\pi}{4}\cos^2\frac{\pi}{4}$$

Example 8 – *Solution*

cont'd

$$\left(\cot\frac{\pi}{4}\right)^2 + \left(\cos\frac{\pi}{4}\right)^2 \stackrel{?}{=} \left(\cot\frac{\pi}{4}\right)^2\left(\cos\frac{\pi}{4}\right)^2$$

$$(1)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 \stackrel{?}{=} (1)^2\left(\frac{\sqrt{2}}{2}\right)^2$$

$$1 + \frac{1}{2} \stackrel{?}{=} 1 \cdot \frac{1}{2}$$

$$\frac{3}{2} \neq \frac{1}{2}$$

Therefore, $\cot^2\theta + \cos^2\theta \neq \cot^2\theta\cos^2\theta$ when $\theta = \pi/4$, so the statement is not an identity.