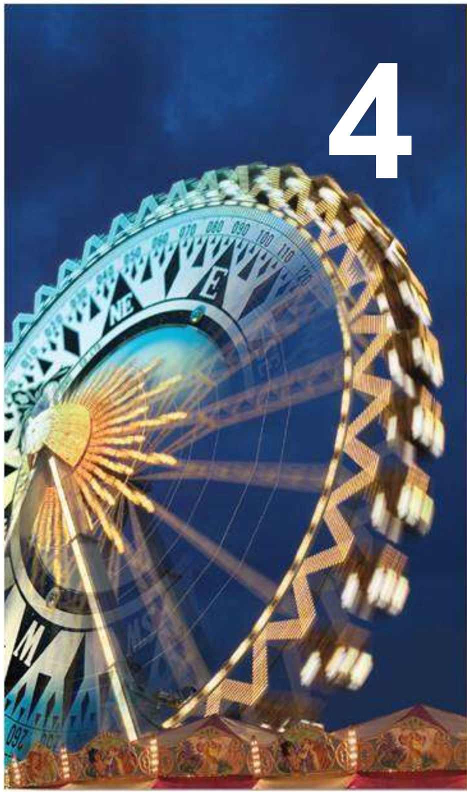


4 Graphing and Inverse Functions

SECTION 4.7

Inverse Trigonometric Functions

Learning Objectives

1. Find the exact value of an inverse trigonometric function.

2. Use a calculator to approximate the value of an inverse trigonometric function.

3. Evaluate a composition involving a trigonometric function and its inverse.

4. Simplify a composition involving a trigonometric and inverse trigonometric function.

Inverse Trigonometric Functions

First, let us review the definition of a function and its inverse.

DEFINITION

A *function* is a rule or correspondence that pairs each element of the domain with exactly one element from the range. That is, a function is a set of ordered pairs in which no two different ordered pairs have the same first coordinate.

The *inverse* of a function is found by interchanging the coordinates in each ordered pair that is an element of the function.

Inverse Trigonometric Functions

INVERSE FUNCTION NOTATION

If $y = f(x)$ is a one-to-one function, then the inverse of f is also a function and can be denoted by $y = f^{-1}(x)$.

Because the graphs of all six trigonometric functions do not pass the horizontal line test, the inverse relations for these functions will not be functions themselves.

Inverse Trigonometric Functions

However, we will see that it is possible to define an inverse that is a function if we restrict the original trigonometric function to certain angles.

In this section, we will limit our discussion of inverse trigonometric functions to the inverses of the three major functions: sine, cosine, and tangent.

The Inverse Sine Relation

The Inverse Sine Relation

To find the inverse of $y = \sin x$, we interchange x and y to obtain

$$x = \sin y$$

This is the equation of the inverse sine relation.

To graph $x = \sin y$, we simply reflect the graph of $y = \sin x$ about the line $y = x$, as shown in Figure 2.

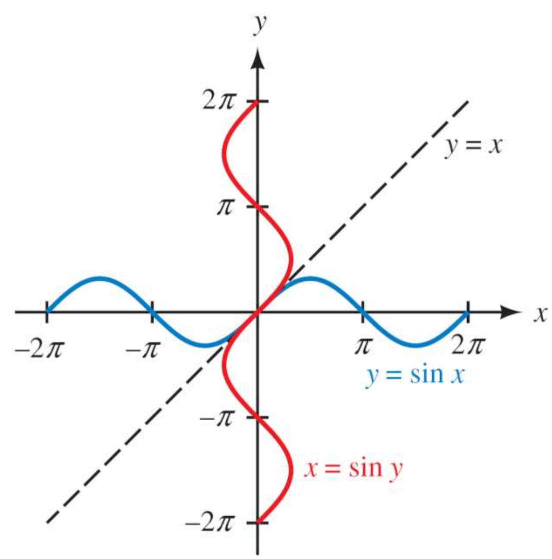


Figure 2

The Inverse Sine Relation

As you can see from the graph, $x = \sin y$ is a relation but not a function.

For every value of x in the domain, there are many values of y.

The graph of $x = \sin y$ fails the vertical line test.

The Inverse Sine Function

The Inverse Sine Function

If the function *y* = sin *x* is to have an inverse that is also a function, it is necessary to restrict the values that *x* can assume so that we may satisfy the horizontal line test.

The interval we restrict it to is $-\pi/2 \le x \le \pi/2$.

The Inverse Sine Function

Figure 3 displays the graph of *y* = sin *x* with the restricted interval showing.

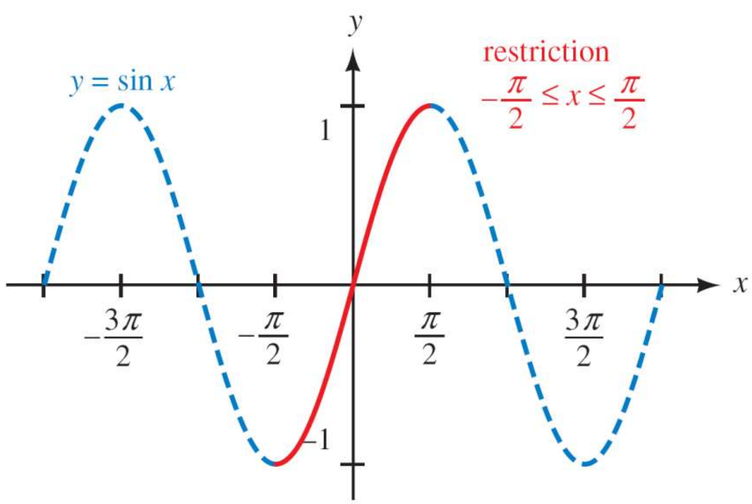


Figure 3

Notice that this segment of the sine graph passes the horizontal line test, and it maintains the full range of the function $-1 \leq y \leq 1$.

The Inverse Sine Function

Figure 4 shows the graph of the inverse relation *x* = sin *y* with the restricted interval after the sine curve has been reflected about the line *y* = *x*.

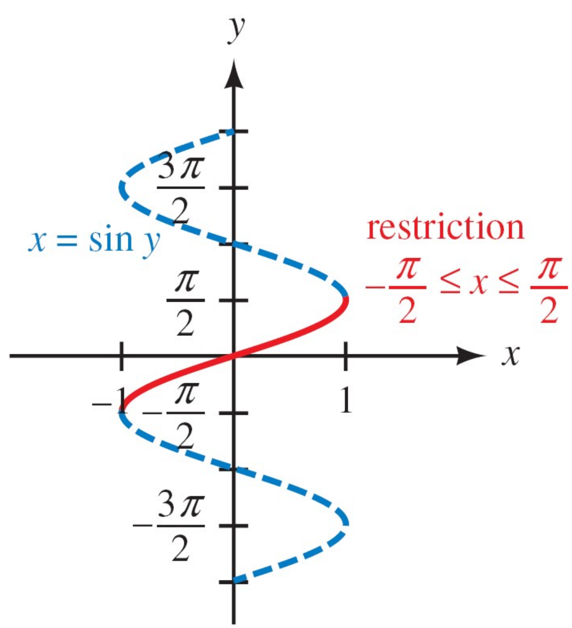


Figure 4

The Inverse Sine Function

It is apparent from Figure 4 that if *x* = sin *y* is restricted to the interval $-\pi/2 \leq y \leq \pi/2,$ then each value of *x* between –1 and 1 is associated with exactly one value of *y*, and we have a function rather than just a relation.

The equation *x* = sin *y*, together with the restriction $-\pi/2 \leq y \leq \pi/2,$ forms the inverse sine function. To designate this function, we use the following notation.

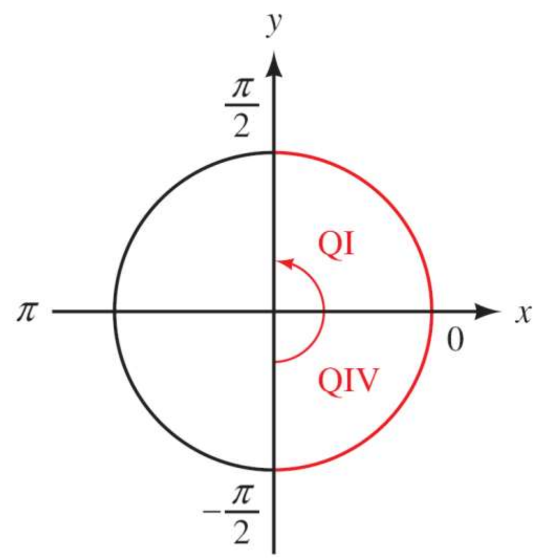
NOTATION

The notation used to indicate the inverse sine function is as follows:

Notation	Meaning
$y = \sin^{-1} x$ or $y = \arcsin x$	$x = \sin y$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

In words: y is the angle between $-\pi/2$ and $\pi/2$, inclusive, whose sine is x.

The Inverse Sine Function



The inverse sine function will return an angle between $-\pi/2 \le x \le \pi/2$ inclusive, corresponding to QIV or QI.

The Inverse Cosine Function

The Inverse Cosine Function

Just as we did for the sine function, we must restrict the values that *x* can assume with the cosine function in order to satisfy the horizontal line test.

The interval we restrict it to is $0 \le x \le \pi$.

Figure 5 shows the graph of *y* = cos *x* with the restricted interval.

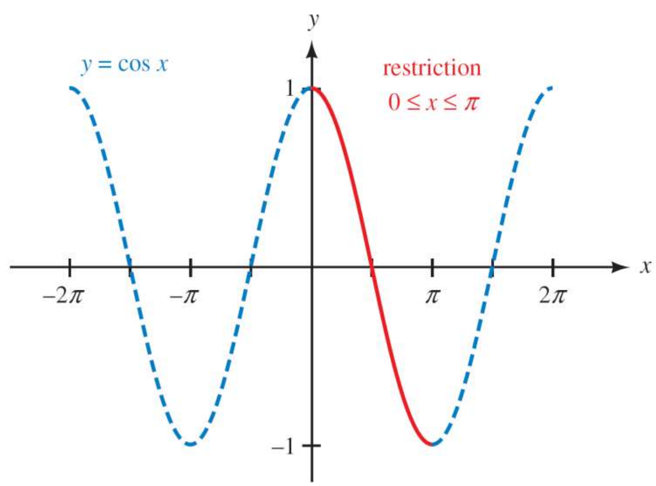


Figure 5

The Inverse Cosine Function

Figure 6 shows the graph of the inverse relation *x* = cos *y* with the restricted interval after the cosine curve has been reflected about the line *y* = *x*.

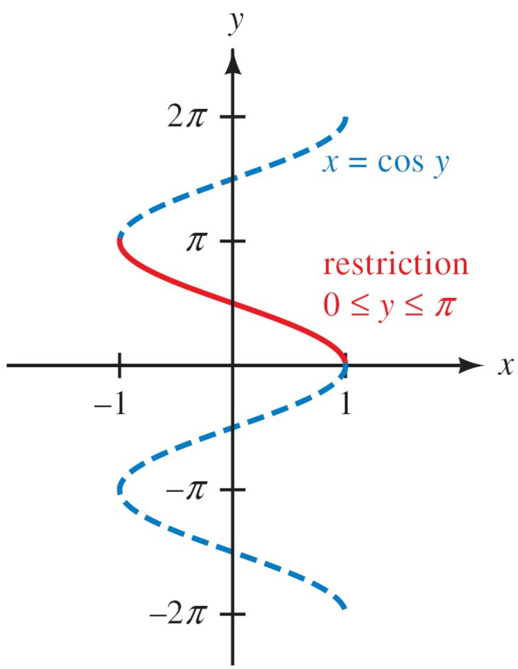


Figure 6

The Inverse Cosine Function

The equation $x = \cos y$, together with the restriction $0 \leq y \leq \pi$, forms the inverse cosine function.

To designate this function we use the following notation.

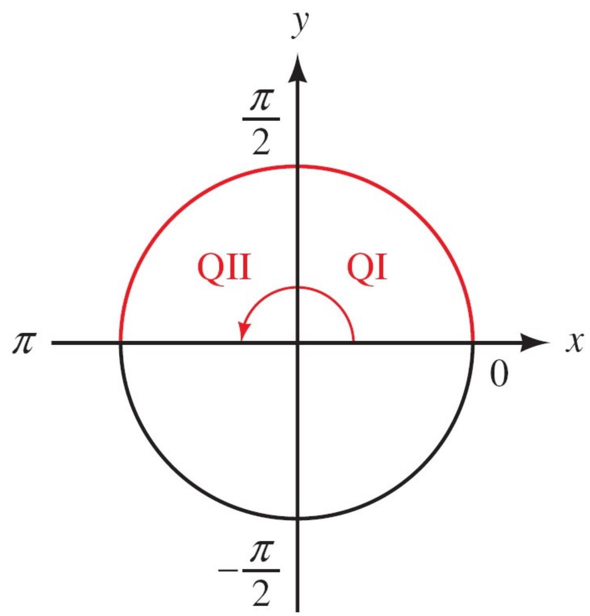
NOTATION

The notation used to indicate the inverse cosine function is as follows:

Notation	**Meaning**
$y = \cos^{-1} x$ or $y = \arccos x$	$x = \cos y$ and $0 \leq y \leq \pi$

In words: y is the angle between 0 and π, inclusive, whose cosine is x.

The Inverse Cosine Function



The inverse cosine function will return an angle between 0 and π, inclusive, corresponding to QI or QII.

The Inverse Tangent Function

The Inverse Tangent Function

For the tangent function, we restrict the values that *x* can assume to the interval $-\pi/2 < x < \pi/2$.

Figure 7 shows the graph of *y* = tan *x* with the restricted interval.

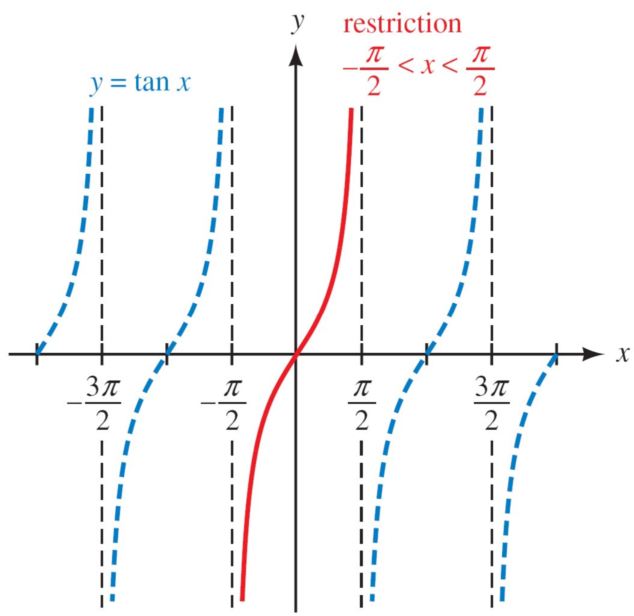


Figure 7

The Inverse Tangent Function

Figure 8 shows the graph of the inverse relation $x = \tan y$ with the restricted interval after it has been reflected about the line $y = x$.

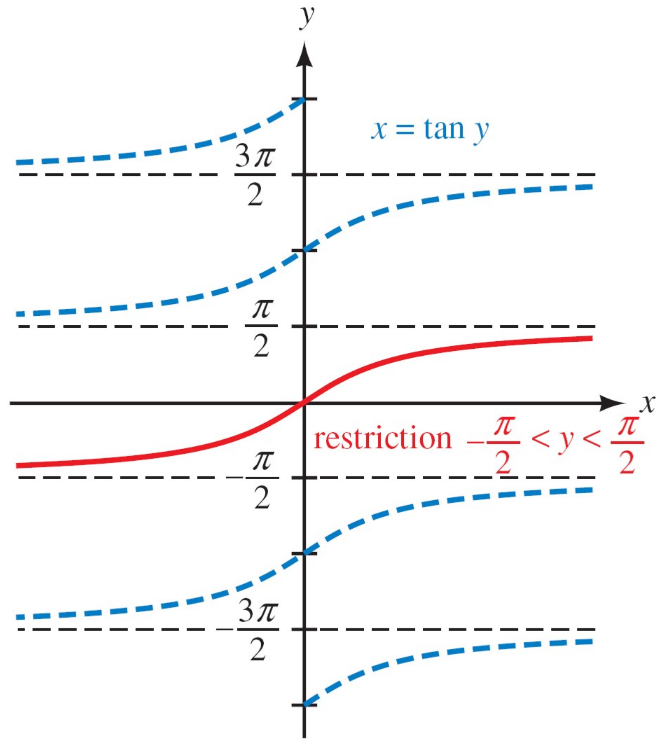


Figure 8

The Inverse Tangent Function

The equation x = tan y, together with the restriction $-\pi/2 < y < \pi/2$, forms the inverse tangent function.

To designate this function we use the following notation.

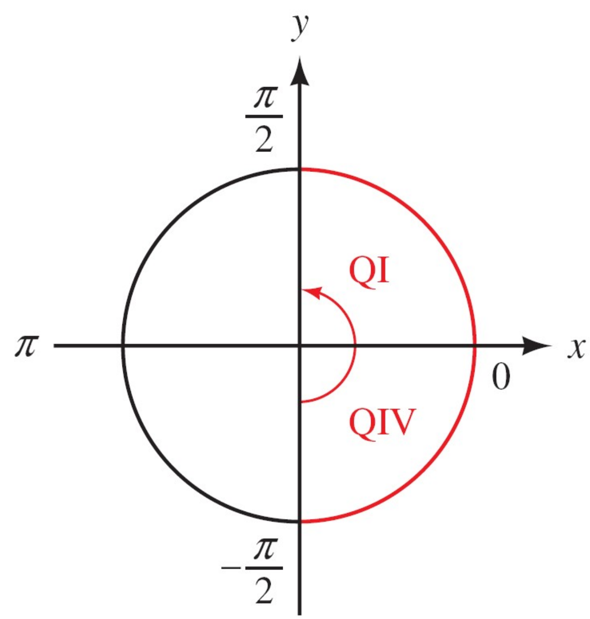
NOTATION

The notation used to indicate the inverse tangent function is as follows:

Notation	Meaning
$y = \tan^{-1} x$ or $y = \arctan x$	$x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

In words: y is the angle between $-\pi/2$ and $\pi/2$ whose tangent is x.

The Inverse Tangent Function

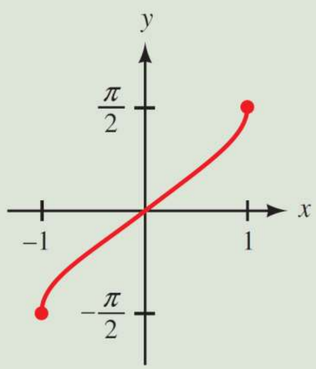
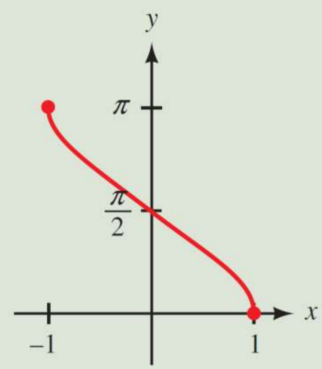
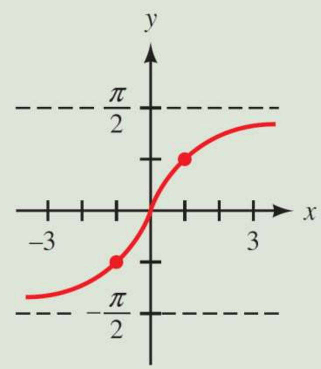


The inverse tangent function will return an angle between $-\pi/2$ and $\pi/2$, corresponding to QIV or QI.

The Inverse Tangent Function

To summarize, here are the three inverse trigonometric functions we have presented, along with the domain, range, and graph for each.

INVERSE TRIGONOMETRIC FUNCTIONS

Inverse Sine	Inverse Cosine	Inverse Tangent
$y = \sin^{-1} x = \arcsin x$	$y = \cos^{-1} x = \arccos x$	$y = \tan^{-1} x = \arctan x$
		
Domain: $-1 \le x \le 1$	Domain: $-1 \le x \le 1$	Domain: all real numbers
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	Range: $0 \le y \le \pi$	Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Example 1

Evaluate in radians without using a calculator or tables.

a. $\sin^{-1}\frac{1}{2}$ **b.** $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ **c.** $\tan^{-1}(-1)$

Solution:

a. The angle between $-\pi/2$ and $\pi/2$ whose sine is $\frac{1}{2}$ is $\pi/6$.

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

Example 1 – *Solution*

cont'd

b. The angle between 0 and π with a cosine of $-\sqrt{3}/2$ is $5\pi/6$.

$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

c. The angle between $-\pi/2$ and $\pi/2$ the tangent of which is –1 is $-\pi/4$.

$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

Example 2

Use a calculator to evaluate each expression to the nearest tenth of a degree.

a. $\arcsin(0.5075)$ **b.** $\arcsin(-0.5075)$ **c.** $\cos^{-1}(0.6428)$

d. $\cos^{-1}(-0.6428)$ **e.** $\arctan(4.474)$ **d.** $\arctan(-4.474)$

Solution:

Make sure the calculator is set to degree mode, and then enter the number and press the appropriate key.

Example 2 – *Solution*

cont'd

Scientific and graphing calculators are programmed so that the restrictions on the inverse trigonometric functions are automatic.

a. $\arcsin(0.5075) = 30.5°$
b. $\arcsin(-0.5075) = -30.5°$
Reference angle 30.5°

c. $\cos^{-1}(0.6428) = 50.0°$
d. $\cos^{-1}(-0.6428) = 130.0°$
Reference angle 50°

e. $\arctan(4.474) = 77.4°$
f. $\arctan(-4.474) = -77.4°$
Reference angle 77.4°

The Inverse Tangent Function

We will now see how to simplify the result of that example further by removing the absolute value symbol.

Example 3

Simplify $3\left|\sec\theta\right|$ if $\theta = \tan^{-1}\frac{x}{3}$ for some real number *x*.

Solution:

Because $\theta = \tan^{-1}\frac{x}{3}$, we know from the definition of the inverse tangent function that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

For any angle θ within this interval, sec θ will be a positive value.

Therefore, $\left|\sec\theta\right| = \sec\theta$ and we can simplify the expression further as

$$3\left|\sec\theta\right| = 3\sec\theta$$

Example 4

Evaluate each expression.

a. $\sin\left(\sin^{-1}\frac{1}{2}\right)$ **b.** $\sin^{-1}(\sin 135°)$

Solution:

a. From Example 1a we know that $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

Therefore,

$$\sin\left(\sin^{-1}\frac{1}{2}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Example 4 – *Solution*

cont'd

b. Because

$$\sin 135° = \frac{\sqrt{2}}{2},\ \sin^{-1}(\sin 135°) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

will be the angle *y*, $-90° \le y \le 90°$, for which $\sin y = \dfrac{\sqrt{2}}{2}$.

The angle satisfying this requirement is *y* = 45°.

So,

$$\sin^{-1}(\sin 135°) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45°$$

Example 7

Write the expression $\sin\ (\cos^{-1} x)$ as an equivalent algebraic expression in *x* only.

Solution:

We let $\theta = \cos^{-1} x$. Then

$$\cos\theta = x = \frac{x}{1} \qquad \text{and} \qquad 0 \le \theta \le \pi$$

Example 7 – *Solution*

cont'd

We can visualize the problem by drawing in standard position with terminal side in either QI or QII (Figure 12).

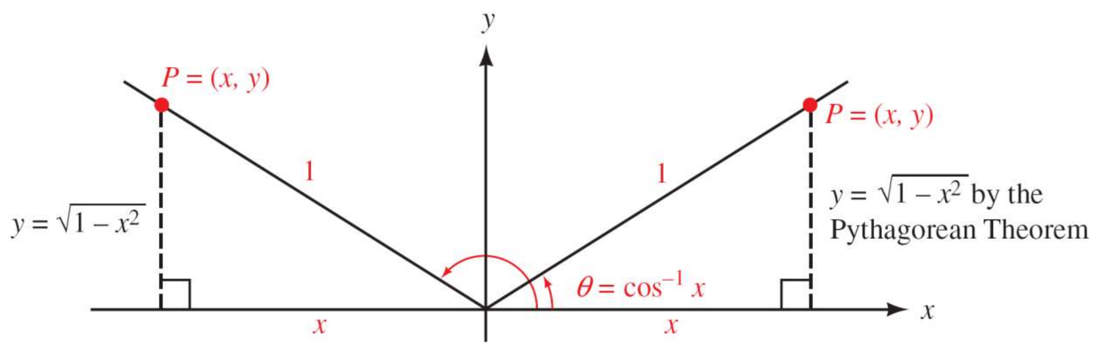


Figure 12

Let *P* = (*x*, *y*) be a point on the terminal side of θ. By Definition I, $\cos\theta = \dfrac{x}{r}$, so *r* must be equal to 1.

Example 7 – *Solution*

cont'd

We can find *y* by applying the Pythagorean Theorem. Notice that *y* will be a positive value in either quadrant.

Because $\sin\theta = \dfrac{y}{r}$,

$$\sin(\cos^{-1} x) = \sin\theta = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}$$

This result is valid whether *x* is positive (θ terminates in QI) or negative (θ terminates in QII).