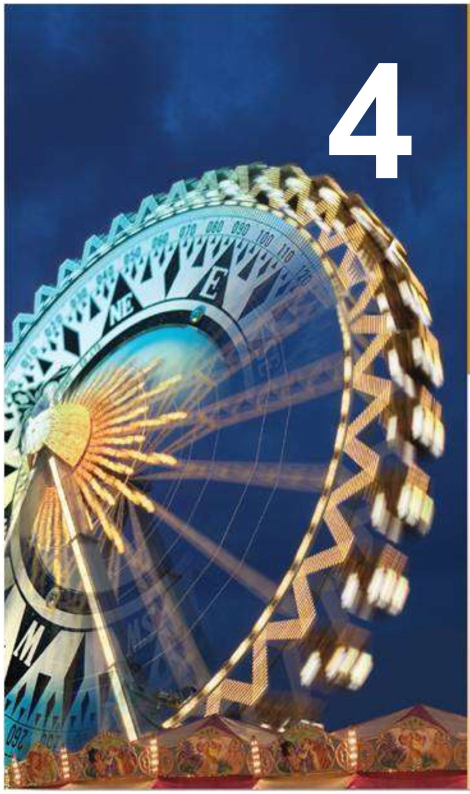

4 Graphing and Inverse Functions

SECTION 4.3

Vertical and Horizontal Translations

Learning Objectives

1. Find the vertical translation of a sine or cosine function.
2. Find the horizontal translation of a sine or cosine function.
3. Identify the phase for a sine or cosine function.
4. Graph a sine or cosine function having a horizontal and vertical translation.

Vertical Translations

Vertical Translations

In general, the graph of $y = f(x) + k$ is the graph of $y = f(x)$ translated k units vertically.

If k is a positive number, the translation is up.

If k is a negative number, the translation is down.

Example 2

The Ferris wheel is shown in Figure 2.

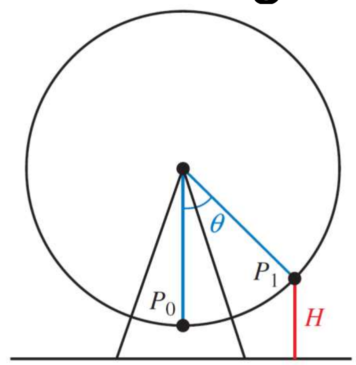

Figure 2

If the diameter of the wheel is 250 feet, the distance from the ground to the bottom of the wheel is 14 feet, and one complete revolution takes 20 minutes, then the height of a rider on a Ferris wheel will be given by the function

$$H = 139 - 125 \cos\left(\frac{\pi}{10}t\right)$$

where t is the number of minutes from the beginning of a ride. Graph a complete cycle of this function.

Example 2 – *Solution*

The term 139 indicates that the cosine graph is shifted 139 units upward. We lightly draw the dashed horizontal line $H = 139$ to act in place of the t-axis, and then proceed as normal.

The amplitude is 125, and there is a reflection about the t-axis due to the negative sign.

One cycle:

$$0 \le \frac{\pi}{10} t \le 2\pi$$

$$0 \le t \le 20 \qquad \text{Multiply by } \frac{10}{\pi}$$

Example 2 – *Solution*

cont'd

The period is 20. Dividing 20 by 4, we will mark the *t*-axis at intervals of 5. Using a rectangle to frame the cycle, we then sketch in the graph as shown in Figure 4.

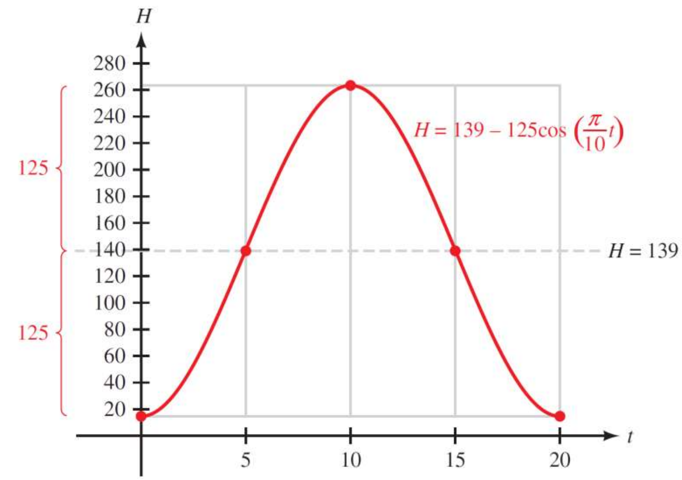

Figure 4

Notice that we measure 125 units above and below the line *H* = 139 to create the frame and that we must remember to plot points for a cycle that is reflected.

Vertical Translations

SUMMARY

The graphs of $y = k + \sin x$ and $y = k + \cos x$ will be sine and cosine graphs that have been translated vertically k units upward if $k > 0$, or k units downward if $k < 0$.

Horizontal Translations

Horizontal Translations

If we add a term to the argument of the function, the graph will be translated in a horizontal direction instead of a vertical direction as demonstrated in the next example.

Example 3

Graph $y = \sin\left(x + \frac{\pi}{2}\right)$, if $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

Solution:

Because we have not graphed an equation of this form before, it is a good idea to begin by making a table (Table 1).

x	$y = \sin\left(x + \frac{\pi}{2}\right)$	(x, y)
$-\frac{\pi}{2}$	$y = \sin\left(-\frac{\pi}{2} + \frac{\pi}{2}\right) = \sin 0 = 0$	$\left(-\frac{\pi}{2}, 0\right)$
0	$y = \sin\left(0 + \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$	$(0, 1)$
$\frac{\pi}{2}$	$y = \sin\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \sin \pi = 0$	$\left(\frac{\pi}{2}, 0\right)$
π	$y = \sin\left(\pi + \frac{\pi}{2}\right) = \sin\frac{3\pi}{2} = -1$	$(\pi, -1)$
$\frac{3\pi}{2}$	$y = \sin\left(\frac{3\pi}{2} + \frac{\pi}{2}\right) = \sin 2\pi = 0$	$\left(\frac{3\pi}{2}, 0\right)$

Table 1

Example 3 – *Solution*

cont'd

In this case, multiples of $\pi/2$ will be the most convenient replacements for x in the table. Also, if we start with $x = -\pi/2$, our first value of y will be 0.

Graphing these points and then drawing the sine curve that connects them gives us the graph of $y = \sin(x + \pi/2)$, as shown in Figure 5.

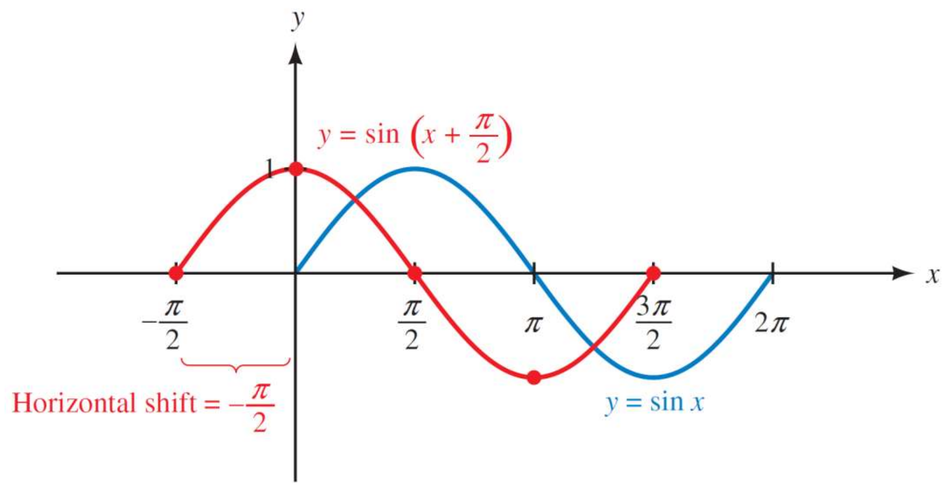

Figure 5

Example 3 – *Solution*

cont'd

It seems that the graph of $y = \sin(x + \pi/2)$ is shifted $\pi/2$ units to the left of the graph of $y = \sin x$.

We say the graph of $y = \sin(x + \pi/2)$ has a *horizontal translation*, or *horizontal shift*, of $-\pi/2$, where the negative sign indicates the shift is to the left (in the negative direction).

Horizontal Translations

We can see why the graph was shifted to the left by looking at how the extra term affects a basic cycle of the sine function.

We know that the sine function completes one cycle when the input value, or argument, varies between 0 and 2π.

One cycle: $0 \le \text{argument} \le 2\pi$

$$0 \le x + \frac{\pi}{2} \le 2\pi \qquad \text{The argument is } x + \frac{\pi}{2}$$

$$-\frac{\pi}{2} \le x \le \frac{3\pi}{2} \qquad \text{Subtract } \frac{\pi}{2} \text{ to isolate } x$$

Horizontal Translations

Notice that a cycle will now begin at $x = -\pi/2$ instead of at zero, and will end at $3\pi/2$ instead of 2π, which agrees with the graph in Figure 5.

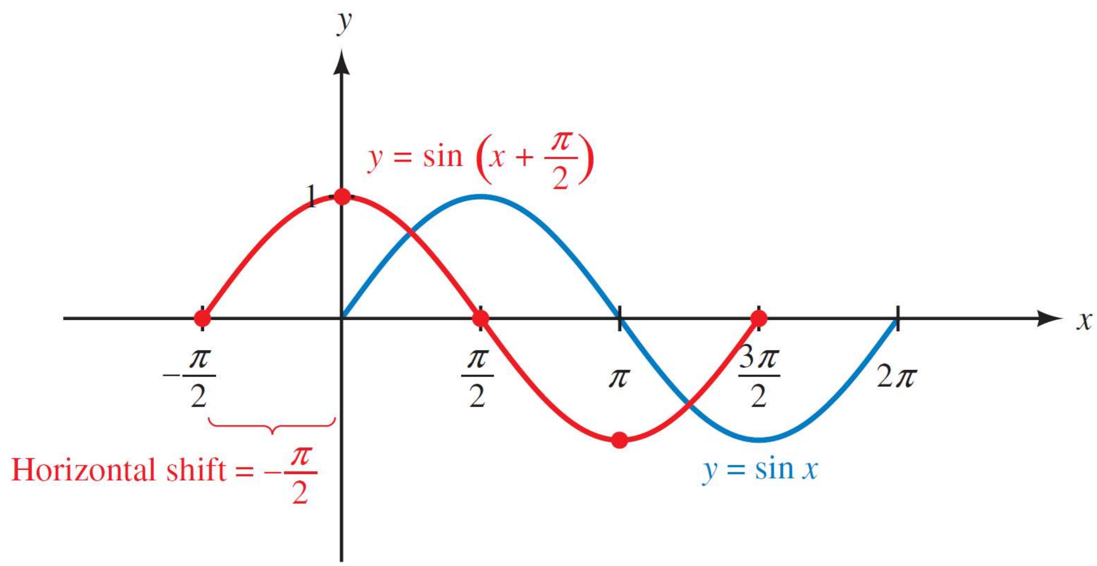

Figure 5

Horizontal Translations

The graph has simply shifted $\pi/2$ units to the left. The horizontal shift is the value of *x* at which the basic cycle begins, which will always be the left value in the above inequality *after x has been isolated*.

SUMMARY

The graphs of $y = \sin(x - h)$ and $y = \cos(x - h)$ will be sine and cosine graphs that have been translated horizontally h units to the right if $h > 0$, or h units to the left if $h < 0$.

Next we look at an example that involves a combination of a period change and a horizontal shift.

Example 5

Graph $y = 4\cos\left(2x - \frac{3\pi}{2}\right)$ for $0 \le x \le 2\pi$.

Solution:

The amplitude is 4. There is no vertical translation because no number has been added to or subtracted from the cosine function.

We determine the period and horizontal translation from a basic cycle.

Example 5 – *Solution*

cont'd

One Cycle: $0 \le 2x - \dfrac{3\pi}{2} \le 2\pi$

$$\frac{3\pi}{2} \le 2x \le \frac{7\pi}{2} \qquad \text{Add } \frac{3\pi}{2} \text{ first}$$

$$\frac{3\pi}{4} \le x \le \frac{7\pi}{4} \qquad \text{Divide by 2}$$

A cycle will begin at $x = 3\pi/4$, so the horizontal shift is $3\pi/4$.

To find the period, we can subtract the left and right endpoints of the cycle

$$\text{Period} = \frac{7\pi}{4} - \frac{3\pi}{4} = \frac{4\pi}{4} = \pi$$

Example 5 – *Solution*

cont'd

or, equivalently, divide 2π by $B = 2$

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

Dividing the period by 4 gives us $\pi/4$. To mark the x-axis, we begin at $x = 3\pi/4$ and add increments of $\pi/4$ as follows:

$$\frac{3\pi}{4} + \frac{\pi}{4} = \frac{4\pi}{4} = \pi \qquad \frac{3\pi}{4} + 2 \cdot \frac{\pi}{4} = \frac{5\pi}{4}$$

$$\frac{3\pi}{4} + 3 \cdot \frac{\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

Example 5 – *Solution*

cont'd

We draw a frame and then sketch the graph of one complete cycle as shown in Figure 7.

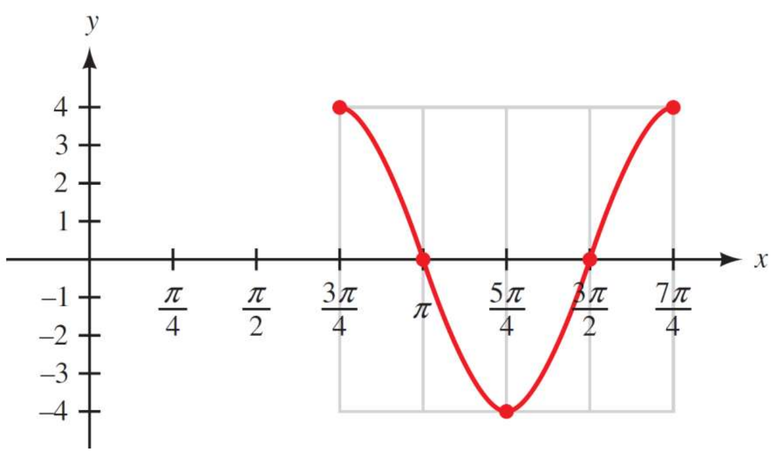

Figure 7

Because the original problem asked for the graph on the interval $0 \leq x \leq 2\pi$, we extend the graph to the right by adding the first quarter of a second cycle.

Example 5 – *Solution*

cont'd

On the left, we add the last three quarters of an additional cycle to reach 0. The final graph is shown in Figure 8.

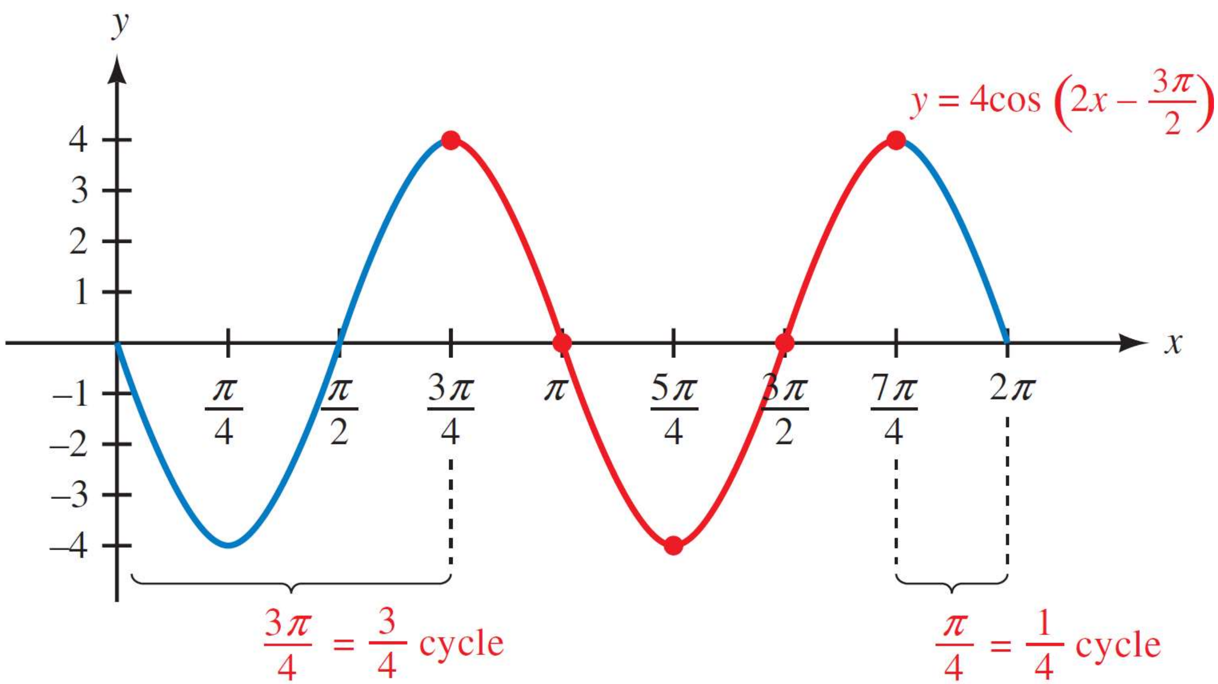

Figure 8

Horizontal Translations

In general, for $y = \sin(Bx + C)$ or $y = \cos(Bx + C)$ to complete one cycle, the quantity $Bx + C$ must vary from 0 to 2π. Therefore, assuming $B > 0$,

$$0 \le Bx + C \le 2\pi \qquad \text{if} \qquad -\frac{C}{B} \le x \le \frac{2\pi - C}{B}$$

The horizontal shift will be the left end point of the cycle, or $-C/B$. If you find the difference between the end points, you will see that the period is $2\pi/B$ as before.

The constant C in $y = \sin(Bx + C)$ or $y = \cos(Bx + C)$ is called the *phase*.

Horizontal Translations

Phase is important in situations, such as when working with alternating currents, where two sinusoidal curves are being compared to one another.

If x represents time, then the phase is the fraction of a standard period of 2π that a point on the graph of $y = \sin(Bx + C)$ lags or leads a corresponding point on the graph of $y = \sin Bx$.

Horizontal Translations

PERIOD, HORIZONTAL SHIFT, AND PHASE FOR SINE AND COSINE

If C is any real number and $B > 0$, then the graphs of $y = \sin(Bx + C)$ and $y = \cos(Bx + C)$ will have

$$\text{Period} = \frac{2\pi}{B} \qquad \text{Horizontal shift} = -\frac{C}{B} \qquad \text{Phase} = C$$

Horizontal Translations

We summarize all the information we have covered about the graphs of the sine and cosine functions.

GRAPHING THE SINE AND COSINE FUNCTIONS

The graphs of $y = k + A\sin(B(x - h))$ and $y = k + A\cos(B(x - h))$, where $B > 0$, will have the following characteristics:

Amplitude $= \lvert A \rvert$	Period $= \dfrac{2\pi}{B}$
Horizontal translation $= h$	Vertical translation $= k$

In addition, if $A < 0$ the graph will be reflected about the x-axis.

Example 6

Graph one complete cycle of $y = 3 - 5\sin\left(\pi x + \frac{\pi}{4}\right)$.

Solution:

First, we rewrite the function by factoring out the coefficient of π.

$$y = 3 - 5\sin\left(\pi x + \frac{\pi}{4}\right) = 3 - 5\sin\left(\pi\left(x + \frac{1}{4}\right)\right)$$

Example 6 – *Solution*

cont'd

In this case, the values are A = –5, B = π, h = –1/4, and k = 3. This gives us

$$\text{Amplitude} = |-5| = 5 \qquad \text{Period} = \frac{2\pi}{\pi} = 2$$

$$\text{Horizontal shift} = -\frac{1}{4} \qquad \text{Vertical shift} = 3$$

Example 6 – *Solution*

cont'd

To verify the period and horizontal shift, and to help sketch the graph, we examine one cycle.

One cycle: $$0 \le \pi x + \frac{\pi}{4} \le 2\pi$$

$$-\frac{\pi}{4} \le \pi x \le \frac{7\pi}{4} \qquad \text{Subtract } \frac{\pi}{4} \text{ first}$$

$$-\frac{1}{4} \le x \le \frac{7}{4} \qquad \text{Divide by } \pi$$

Example 6 – *Solution*

cont'd

Dividing the period by 4 gives $\frac{1}{2}$, so we will mark the *x*-axis in increments of $\frac{1}{2}$ starting with $x = -\frac{1}{4}$. Notice that our frame for the cycle (Figure 9) has been shifted upward 3 units, and we have plotted the key points to account for the *x*-axis reflection.

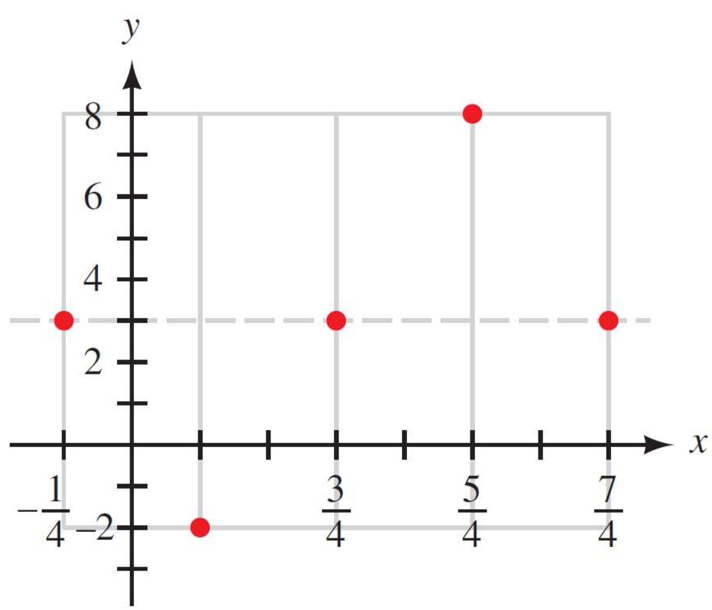

Figure 9

Example 6 – *Solution*

cont'd

The graph is shown in Figure 10.

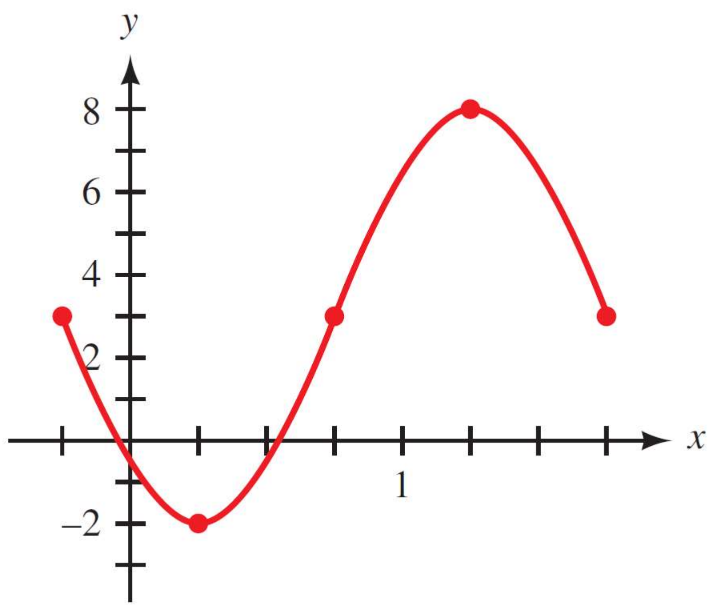

Figure 10