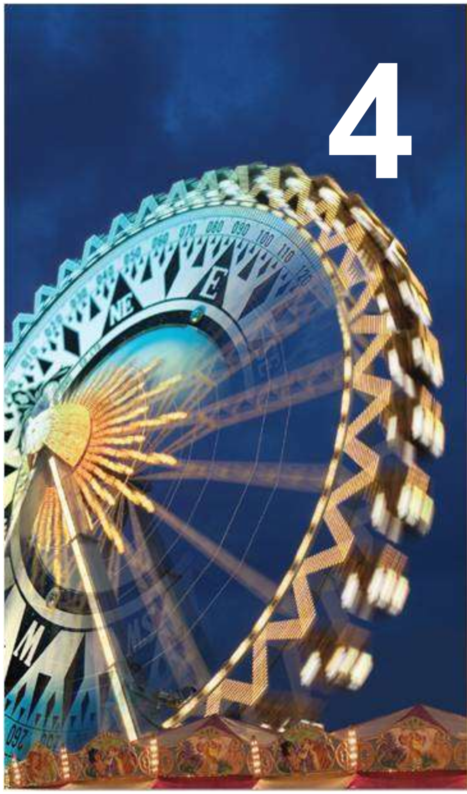


4 Graphing and Inverse Functions

SECTION 4.1

Basic Graphs

Learning Objectives

1. Sketch the graph of a basic trigonometric function.
2. Analyze the graph of a trigonometric function.
3. Evaluate a trigonometric function using the even and odd function relationships.
4. Prove an equation is an identity.

The Sine Graph

The Sine Graph

To graph the function $y = \sin x$, we begin by making a table of values of x and y that satisfy the equation (Table 1), and then use the information in the table to sketch the graph.

x	$y = \sin x$
0	$\sin 0 = 0$
$\frac{\pi}{4}$	$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	$\sin \frac{\pi}{2} = 1$
$\frac{3\pi}{4}$	$\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}$
π	$\sin \pi = 0$
$\frac{5\pi}{4}$	$\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	$\sin \frac{3\pi}{2} = -1$
$\frac{7\pi}{4}$	$\sin \frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$
2π	$\sin 2\pi = 0$

Table 1

The Sine Graph

Graphing each ordered pair and then connecting them with a smooth curve, we obtain the graph in Figure 1:

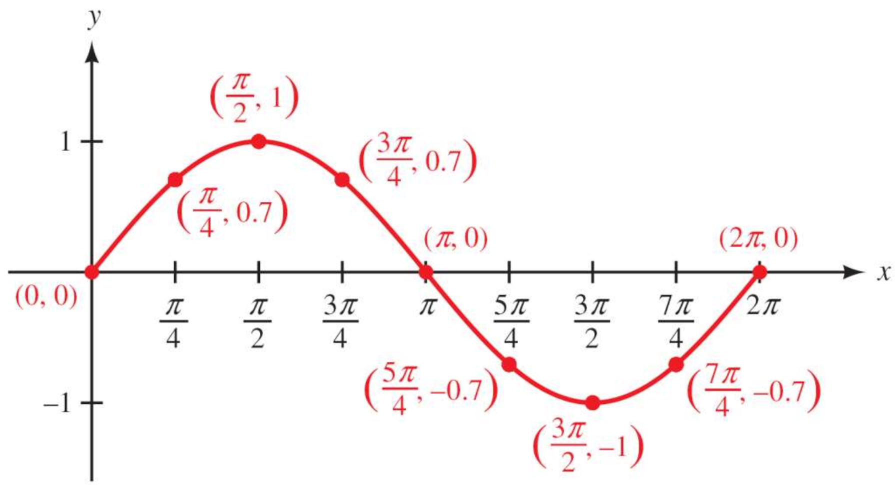


Figure 1

Graphing $y = \sin x$ Using the Unit Circle

Graphing y = sin x Using the Unit Circle

We can also obtain the graph of the sine function by using the unit circle definition (Definition III).

DEFINITION III ■ CIRCULAR FUNCTIONS

If (x, y) is any point on the unit circle, and t is the distance from $(1, 0)$ to (x, y) along the circumference of the unit circle (Figure 4), then,

$$\cos t = x$$

$$\sin t = y$$

$$\tan t = \frac{y}{x} \quad (x \neq 0)$$

$$\cot t = \frac{x}{y} \quad (y \neq 0)$$

$$\csc t = \frac{1}{y} \quad (y \neq 0)$$

$$\sec t = \frac{1}{x} \quad (x \neq 0)$$

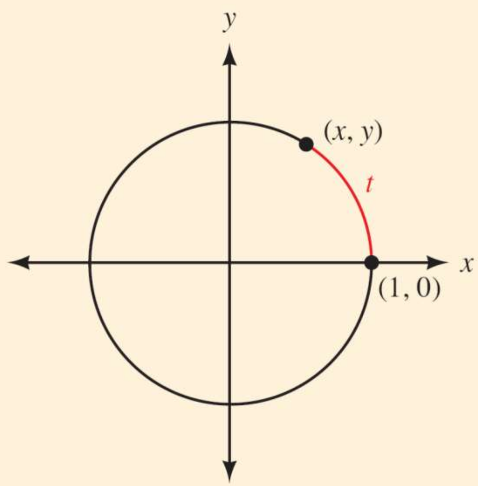


Figure 4

Graphing $y = \sin x$ Using the Unit Circle

Figure 2 shows a diagram of the unit circle.

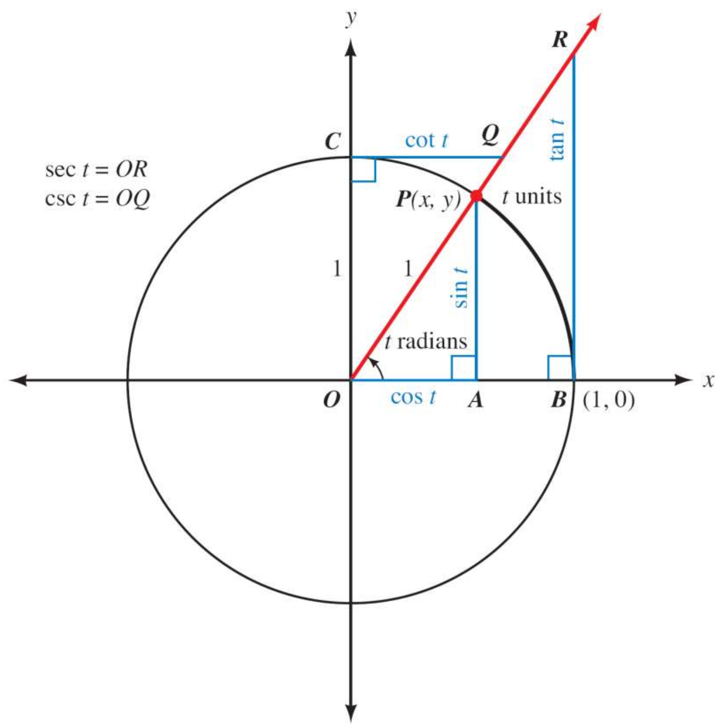


Figure 2

If the point (x, y) is t units from $(1, 0)$ along the circumference of the unit circle, then $\sin t = y$.

Graphing $y = \sin x$ Using the Unit Circle

Therefore, if we start at the point (1, 0) and travel once around the unit circle (a distance of 2π units), we can find the value of y in the equation $y = \sin t$ by simply keeping track of the y-coordinates of the points that are t units from (1, 0).

As t increases from 0 to $\pi/2$, meaning P travels from (1, 0) to (0, 1), $y = \sin t$ increases from 0 to 1.

As t continues in QII from $\pi/2$ to π, y decreases from 1 back to 0.

Graphing $y = \sin x$ Using the Unit Circle

In QIII the length of segment *AP* increases from 0 to 1, but because it is located below the *x*-axis the *y*-coordinate is negative.

So, as *t* increases from π to $3\pi/2$, *y* decreases from 0 to –1.

Finally, as *t* increases from $3\pi/2$ to 2π in QIV, bringing *P* back to (1, 0), *y* increases from –1 back to 0.

Graphing y = sin x Using the Unit Circle

Figure 3 illustrates how the *y*-coordinate of *P* (or *AP*) is used to construct the graph of the sine function as *t* increases.

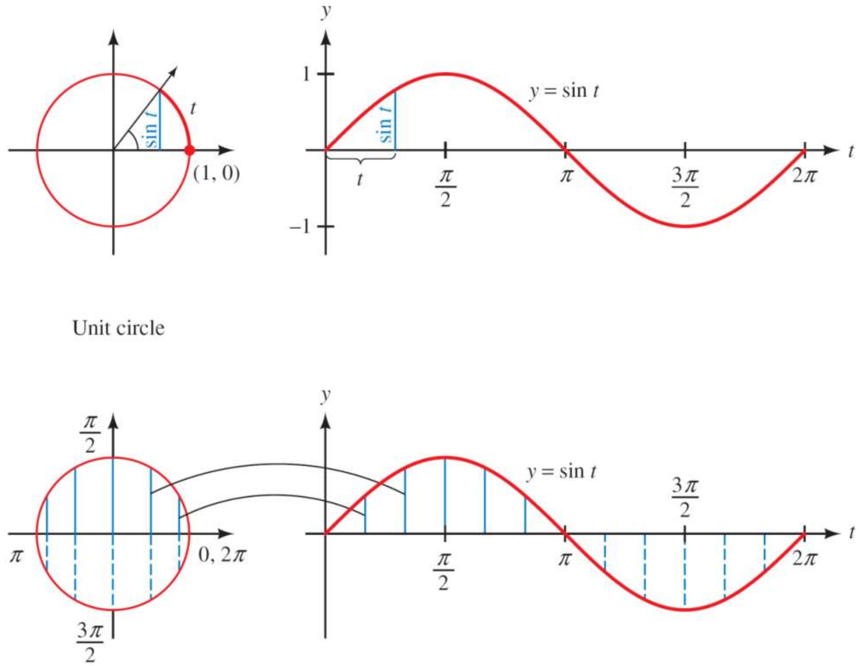


Figure 3

Extending the Sine Graph

Extending the Sine Graph

Figures 1 and 3 each show one complete cycle of *y* = sin *x*.

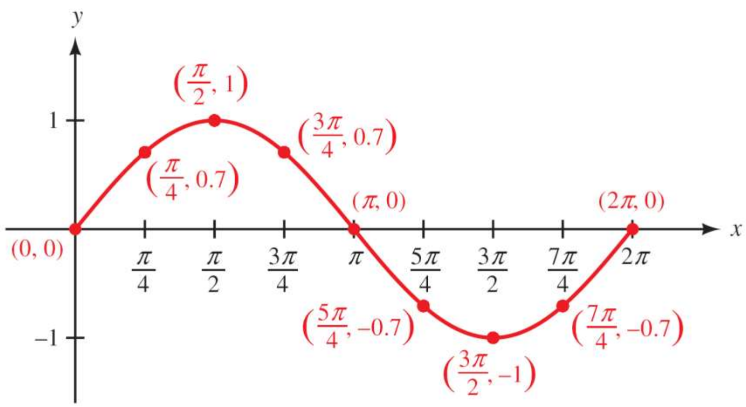


Figure 1

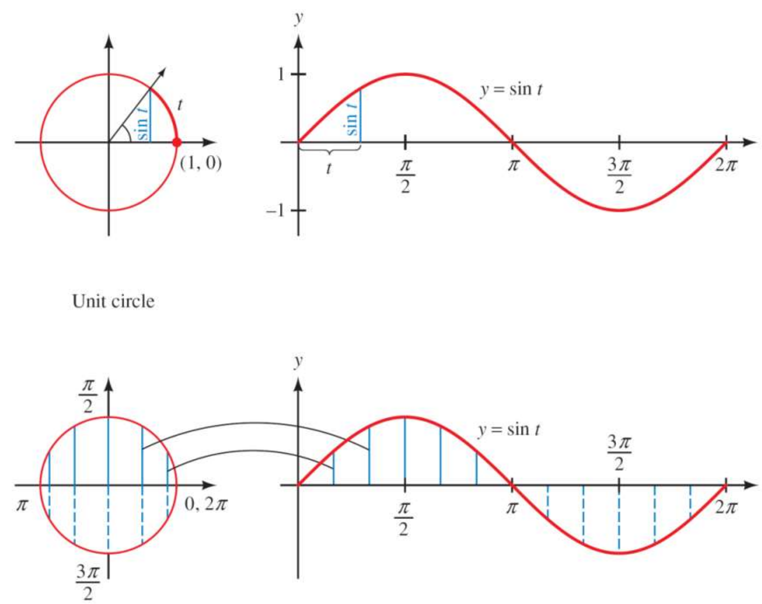


Figure 3

Extending the Sine Graph

Figure 4 shows the graph of $y = \sin x$ extended beyond the interval from $x = 0$ to $x = 2\pi$.

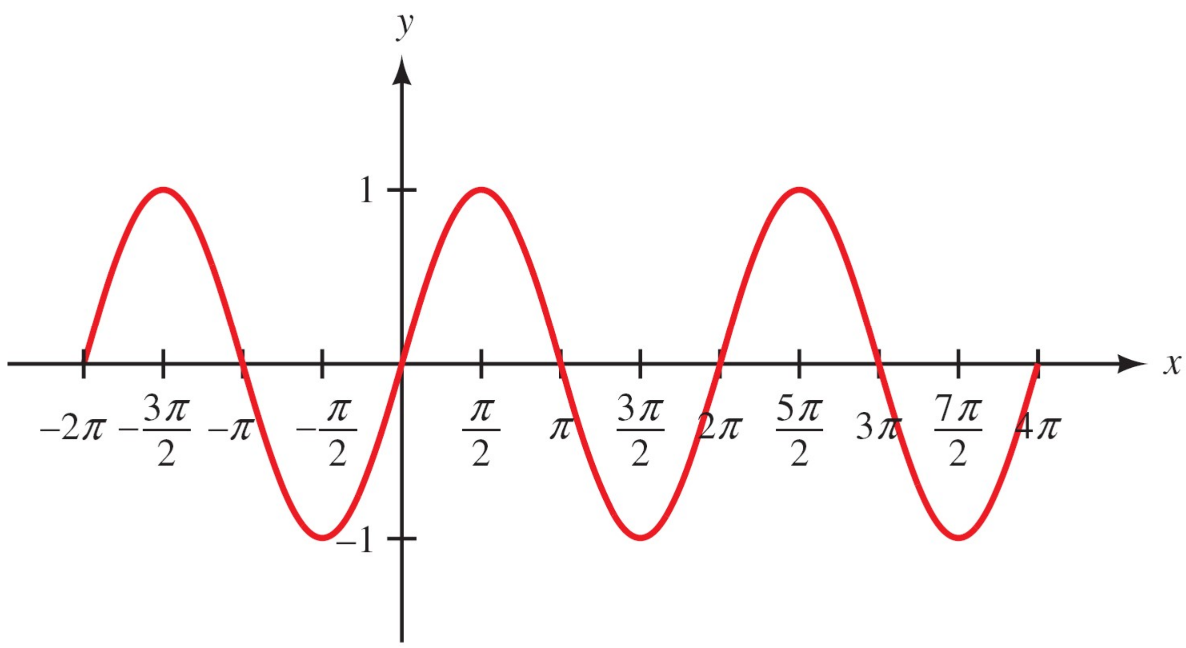


Figure 4

Extending the Sine Graph

The graph of $y = \sin x$ never goes above 1 or below –1, repeats itself every 2π units on the x-axis, and crosses the x-axis at multiples of π. This gives rise to the following three definitions.

DEFINITION ■ PERIOD

For any function $y = f(x)$, the smallest positive number p for which

$$f(x + p) = f(x)$$

for all x in the domain of f is called the *period* of $f(x)$.

Extending the Sine Graph

In the case of $y = \sin x$, the period is 2π because $p = 2\pi$ is the smallest positive number for which $\sin (x + p) = \sin x$ for all x.

DEFINITION ■ AMPLITUDE

If the greatest value of y is M and the least value of y is m, then the *amplitude* of the graph of y is defined to be

$$A = \frac{1}{2}|M - m|$$

Extending the Sine Graph

In the case of $y = \sin x$, the amplitude is 1 because

$$\frac{1}{2}\left|1 - (-1)\right| = \frac{1}{2}(2) = 1$$

DEFINITION ■ ZERO

A *zero* of a function $y = f(x)$ is any domain value $x = c$ for which $f(c) = 0$. If c is a real number, then $x = c$ will be an x-intercept of the graph of $y = f(x)$.

From the graph of $y = \sin x$, we see that the sine function has an infinite number of zeros, which are the values $x = k\pi$ for any integer k.

Extending the Sine Graph

We have known that the domain for the sine function is all real numbers. Because point *P* in Figure 2 must be on the unit circle, we have

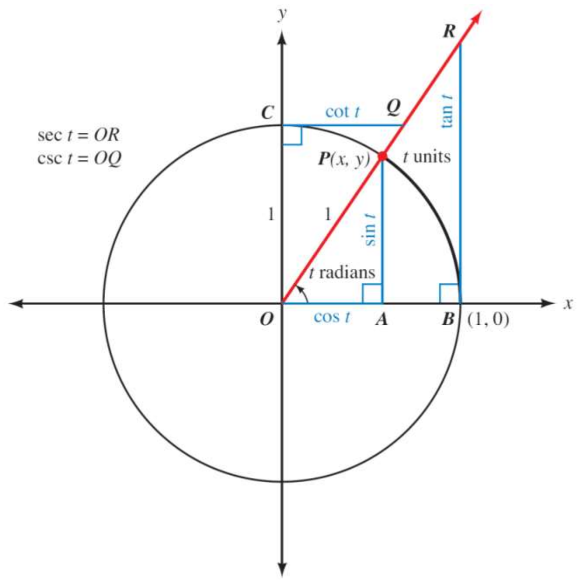


Figure 2

$$-1 \leq y \leq 1 \qquad \text{which implies} \qquad -1 \leq \sin t \leq 1$$

This means the sine function has a range of [–1, 1]. The sine of any angle can only be a value between –1 and 1, inclusive.

The Cosine Graph

The Cosine Graph

The graph of $y = \cos x$ has the same general shape as the graph of $y = \sin x$.

Example 1

Sketch the graph of y = cos x.

Solution:

We can arrive at the graph by making a table of convenient values of x and y (Table 2).

x	$y = \cos x$
0	$\cos 0 = 1$
$\frac{\pi}{4}$	$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	$\cos \frac{\pi}{2} = 0$
$\frac{3\pi}{4}$	$\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$
π	$\cos \pi = -1$
$\frac{5\pi}{4}$	$\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	$\cos \frac{3\pi}{2} = 0$
$\frac{7\pi}{4}$	$\cos \frac{7\pi}{4} = \frac{\sqrt{2}}{2}$
2π	$\cos 2\pi = 1$

Table 2

Example 1 – *Solution*

cont'd

Plotting points, we obtain the graph shown in Figure 5.

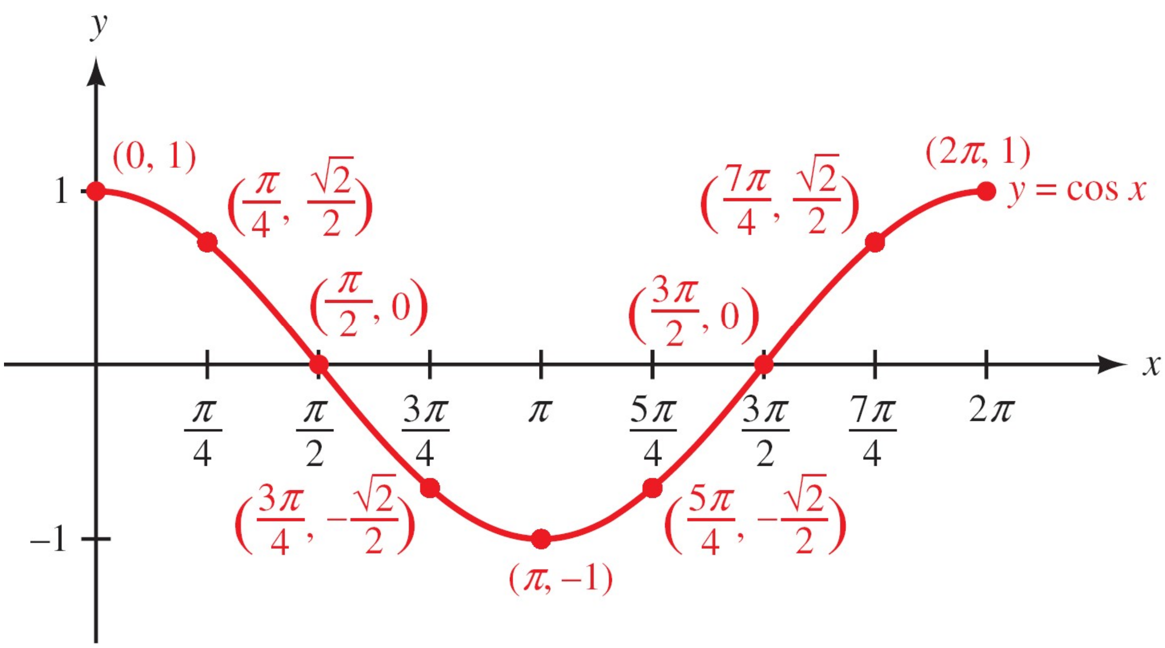


Figure 5

The Cosine Graph

We can generate the graph of the cosine function using the unit circle just as we did for the sine function.

By Definition III, if the point (x, y) is t units from (1, 0) along the circumference of the unit circle, then $\cos t = x$.

DEFINITION III ■ CIRCULAR FUNCTIONS

If (x, y) is any point on the unit circle, and t is the distance from (1, 0) to (x, y) along the circumference of the unit circle (Figure 4), then,

$$\cos t = x$$

$$\sin t = y$$

$$\tan t = \frac{y}{x} \quad (x \neq 0)$$

$$\cot t = \frac{x}{y} \quad (y \neq 0)$$

$$\csc t = \frac{1}{y} \quad (y \neq 0)$$

$$\sec t = \frac{1}{x} \quad (x \neq 0)$$

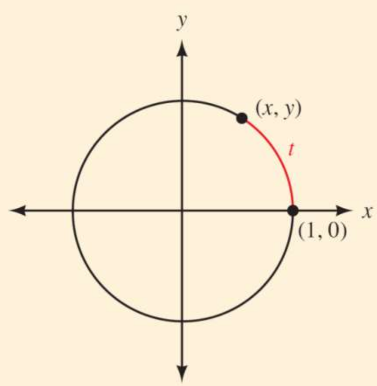


Figure 4

The Cosine Graph

We start at the point (1, 0) and travel once around the unit circle, keeping track of the x-coordinates of the points that are t units from (1, 0).

The Cosine Graph

To help visualize how the *x*-coordinates generate the cosine graph, we have rotated the unit circle 90° counterclockwise so that we may represent the *x*-coordinates as vertical line segments (Figure 6).

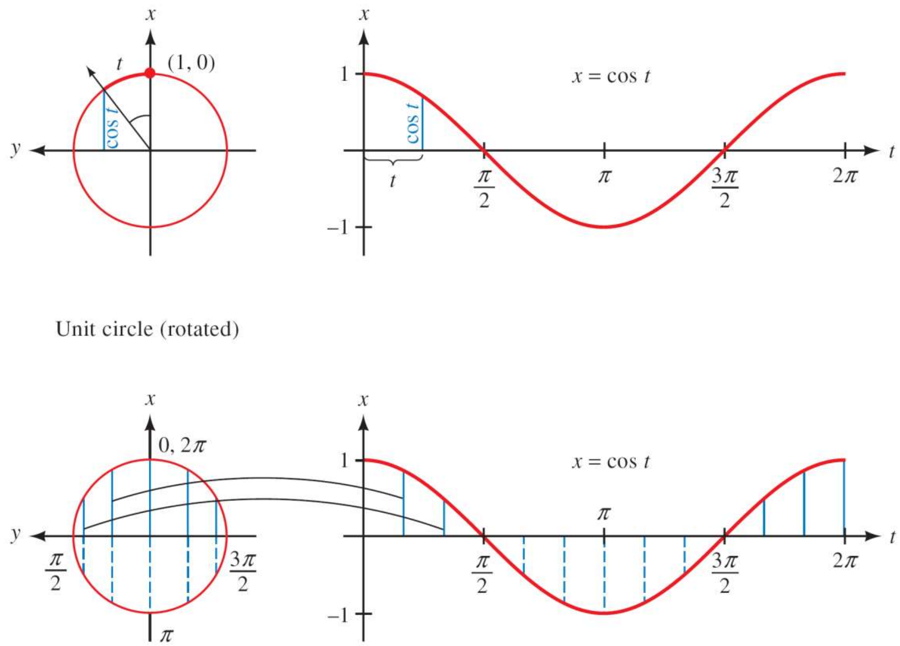


Figure 6

The Cosine Graph

Extending this graph to the right of 2π and to the left of 0, we obtain the graph shown in Figure 7.

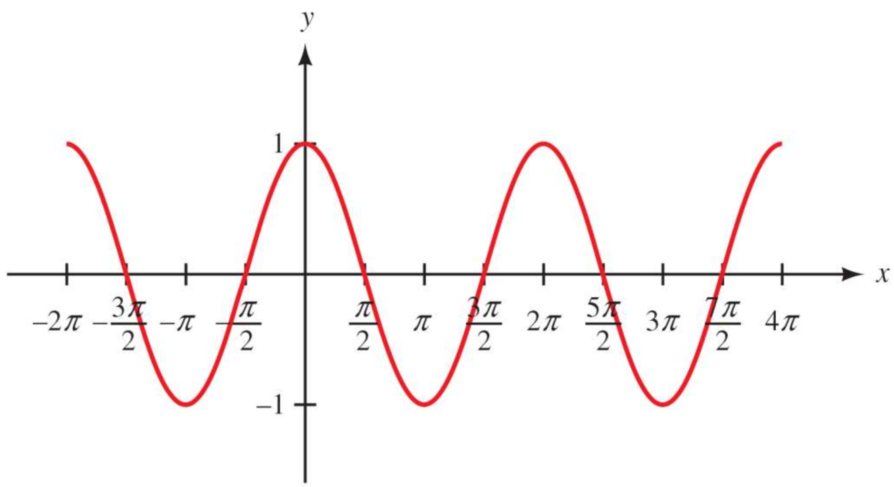


Figure 7

As this figure indicates, the period, amplitude, and range of the cosine function are the same as for the sine function. The zeros, or *x*-intercepts, of *y* = cos *x* are the values $x = \frac{\pi}{2} + k\pi$ for any integer *k*.

The Tangent Graph

The Tangent Graph

Table 3 lists some solutions to the equation $y = \tan x$ between $x = 0$ and $x = \pi$.

x	$\tan x$
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3} \approx 1.7$
$\frac{\pi}{2}$	undefined
$\frac{2\pi}{3}$	$-\sqrt{3} \approx -1.7$
$\frac{3\pi}{4}$	-1
π	0

Table 3

The Tangent Graph

We know that the tangent function will be undefined at $x = \pi/2$ because of the division by zero. Figure 9 shows the graph based on the information from Table 3.

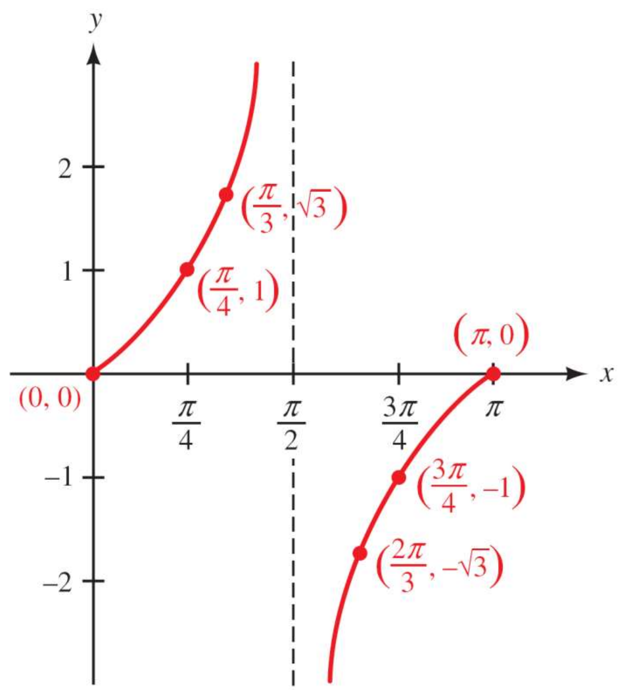


Figure 9

The Tangent Graph

Because y = tan x is undefined at $x = \pi/2$, there is no point on the graph with an x-coordinate of $\pi/2$.

To help us remember this, we have drawn a dotted vertical line through $x = \pi/2$. This vertical line is called an *asymptote*.

The graph will never cross or touch this line.

The Tangent Graph

If we were to calculate values of tan *x* when *x* is very close to $\pi/2$ (or very close to 90° in degree mode), we would find that tan *x* would become very large for values of *x* just to the left of the asymptote and very large in the negative direction for values of *x* just to the right of the asymptote, as shown in Table 4.

x	$\tan x$
85°	11.4
89°	57.3
89.9°	573.0
89.99°	5729.6
90.01°	−5729.6
90.1°	−573.0
91°	−57.3
95°	−11.4

Table 4

The Tangent Graph

Extending the graph in Figure 9 to the right of π and to the left of 0, we obtain the graph shown in Figure 11. As this figure indicates, the period of $y = \tan x$ is π.

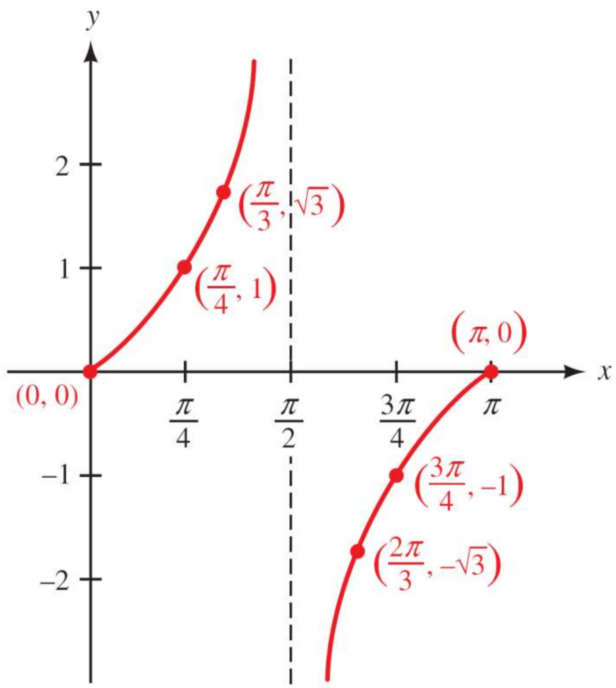


Figure 9

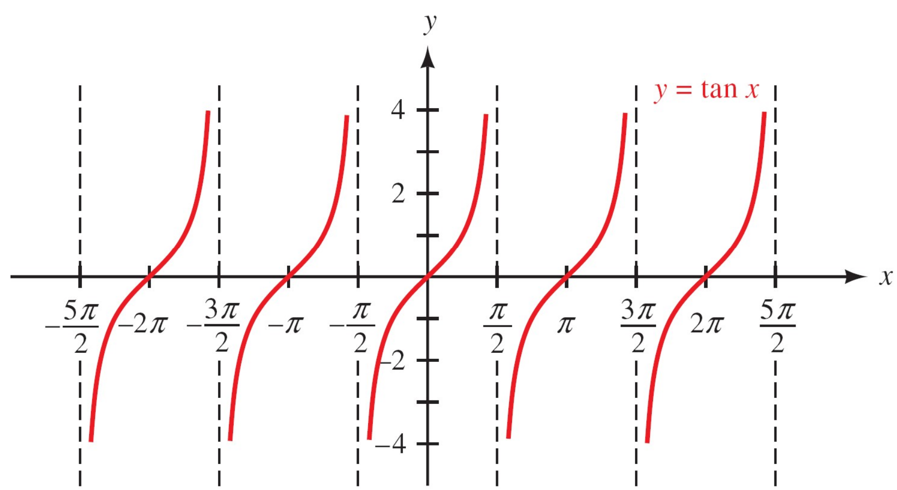


Figure 11

The Tangent Graph

The tangent function has no amplitude because there is no largest or smallest value of y on the graph of $y = \tan x$. For this same reason, the range of the tangent function is all real numbers.

Because $\tan x = \sin x/\cos x$, the zeros for the tangent function are the same as for the sine; that is, $x = k\pi$ for any integer k.

The vertical asymptotes correspond to the zeros of the cosine function, which are $x = \pi/2 + k\pi$ for any integer k.

The Cosecant Graph

The Cosecant Graph

Now that we have the graph of the sine, cosine, and tangent functions, we can use the reciprocal identities to sketch the graph of the remaining three trigonometric functions.

Example 2

Sketch the graph of $y = \csc x$.

Solution:

To graph $y = \csc x$, we can use the fact that $\csc x$ is the reciprocal of $\sin x$. In Table 5, we use the values of $\sin x$ from Table 1 and take reciprocals.

x	$y = \sin x$
0	$\sin 0 = 0$
$\frac{\pi}{4}$	$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	$\sin \frac{\pi}{2} = 1$
$\frac{3\pi}{4}$	$\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}$
π	$\sin \pi = 0$
$\frac{5\pi}{4}$	$\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	$\sin \frac{3\pi}{2} = -1$
$\frac{7\pi}{4}$	$\sin \frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$
2π	$\sin 2\pi = 0$

Table 1

x	$\sin x$	$\csc x = 1/\sin x$
0	0	undefined
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2} \approx 1.4$
$\frac{\pi}{2}$	1	1
$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2} \approx 1.4$
π	0	undefined
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\sqrt{2} \approx -1.4$
$\frac{3\pi}{2}$	-1	-1
$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\sqrt{2} \approx -1.4$
2π	0	undefined

Table 5

Example 2 – *Solution*

cont'd

Filling in with some additional points, we obtain the graph shown in Figure 12.

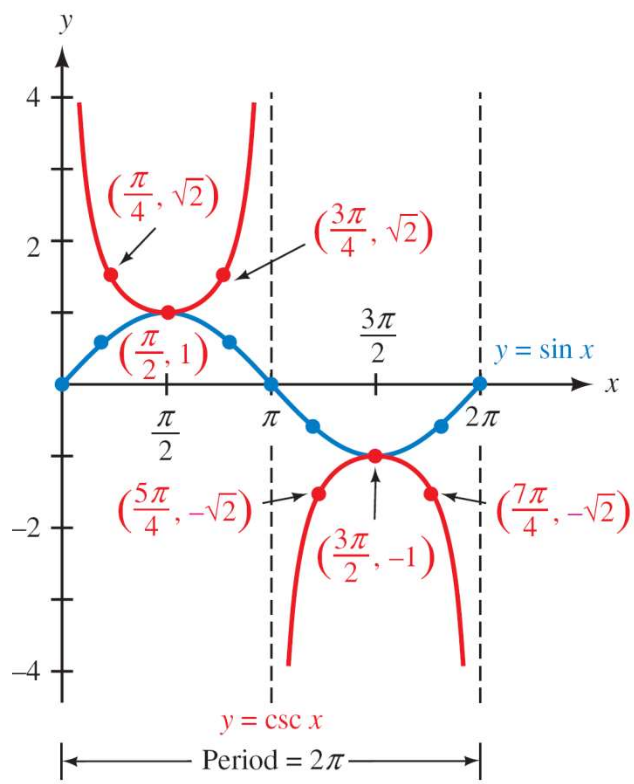


Figure 12

The Cosecant Graph

The cosecant function will be undefined whenever the sine function is zero, so that $y = \csc x$ has vertical asymptotes at the values $x = k\pi$ for any integer k.

Because $y = \sin x$ repeats every 2π, so do the reciprocals of $\sin x$, so the period of $y = \csc x$ is 2π. As was the case with $y = \tan x$, there is no amplitude.

The range of $y = \csc x$ is $y \leq -1$ or $y \geq 1$, or in interval notation, $(-\infty, -1] \cup [1, \infty)$. The cosecant function has no zeros because y cannot ever be equal to zero. Notice in Figure 12 that the graph of $y = \csc x$ never crosses the x-axis.

The Cotangent and Secant Graphs

The Cotangent and Secant Graphs

The graphs of y = cot x and y = sec x are shown in Figures 16 and 17 respectively.

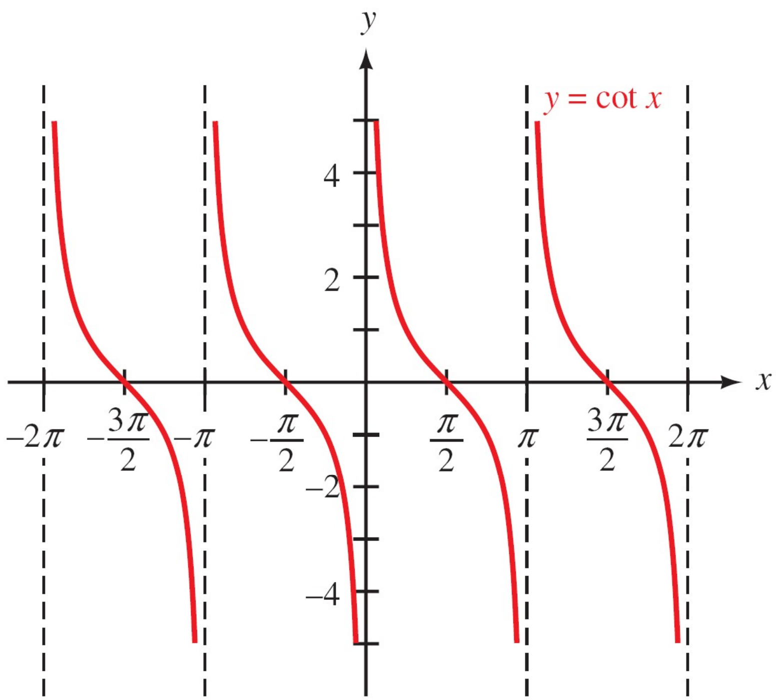


Figure 16

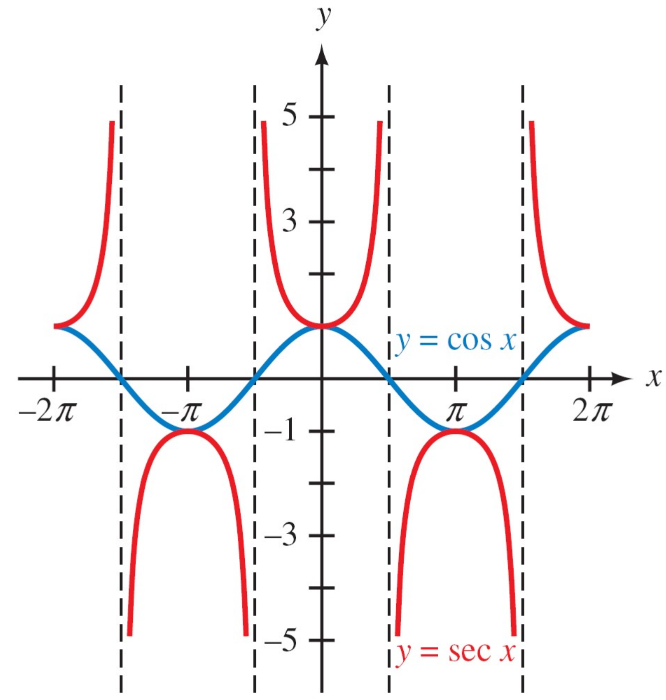
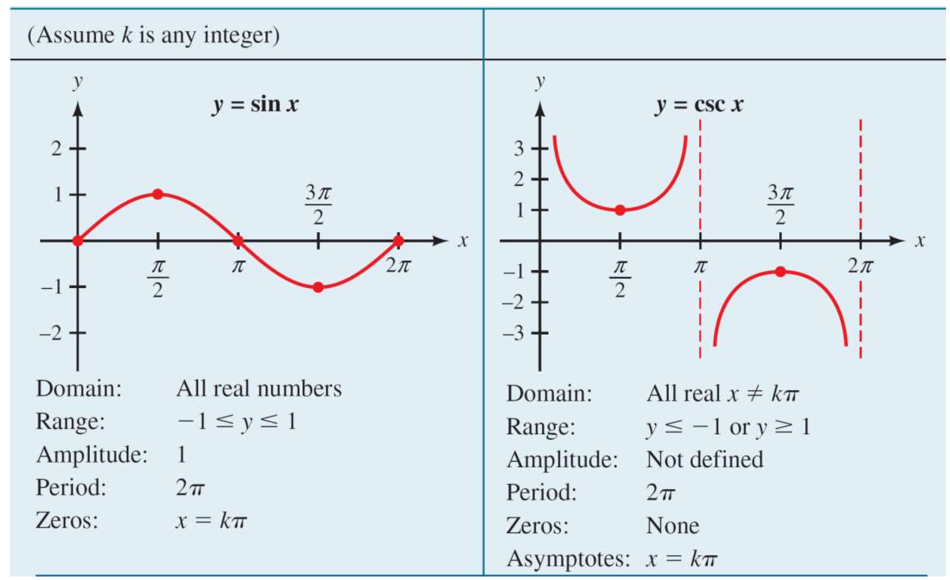


Figure 17

The Cotangent and Secant Graphs

Table 6 is a summary of the important facts associated with the graphs of our trigonometric functions.

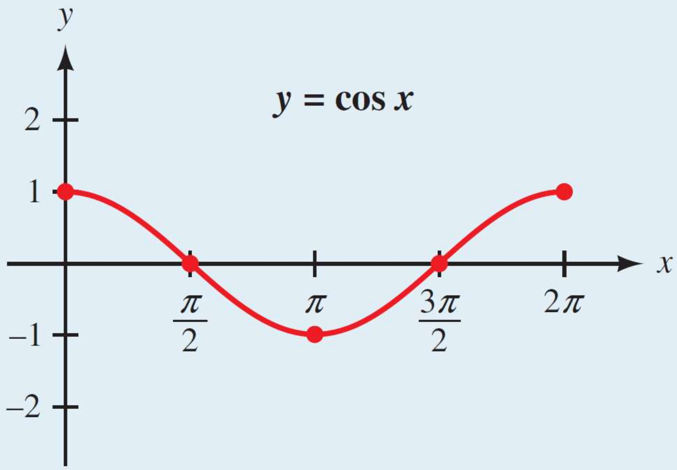
Each graph shows one cycle for the corresponding function, which we will refer to as the *basic cycle*. Keep in mind that all these graphs repeat indefinitely to the left and to the right.



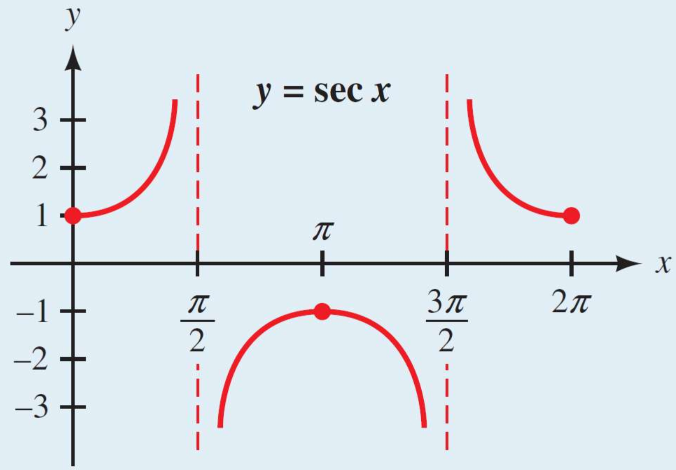
Graphs of the Trigonometric Functions

Table 6

The Cotangent and Secant Graphs



Domain:	All real numbers
Range:	$-1 \le y \le 1$
Amplitude:	1
Period:	2π
Zeros:	$x = \dfrac{\pi}{2} + k\pi$

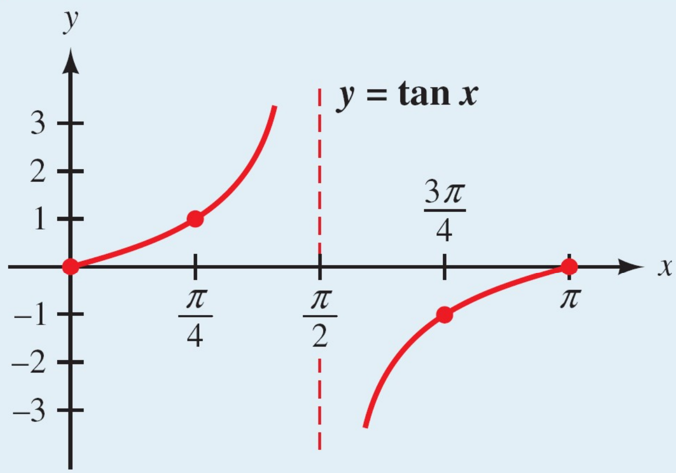


Domain:	All real $x \ne \dfrac{\pi}{2} + k\pi$
Range:	$y \le -1$ or $y \ge 1$
Amplitude:	Not defined
Period:	2π
Zeros:	None
Asymptotes:	$x = \dfrac{\pi}{2} + k\pi$

Graphs of the Trigonometric Functions

Table 6(continued)

The Cotangent and Secant Graphs



Domain: All real $x \neq \frac{\pi}{2} + k\pi$

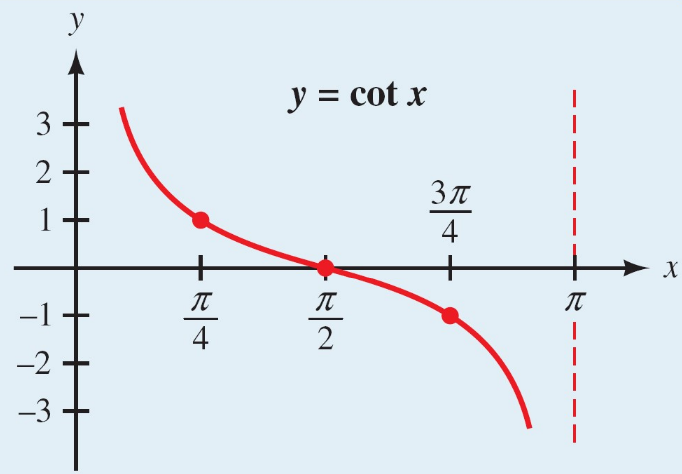
Range: All real numbers

Amplitude: Not defined

Period: π

Zeros: $x = k\pi$

Asymptotes: $x = \frac{\pi}{2} + k\pi$



Domain: All real $x \neq k\pi$

Range: All real numbers

Amplitude: Not defined

Period: π

Zeros: $x = \frac{\pi}{2} + k\pi$

Asymptotes: $x = k\pi$

Graphs of the Trigonometric Functions

Table 6(continued)

Even and Odd Functions

Even and Odd Functions

DEFINITION

An *even function* is a function for which

$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$

The graph of an even function is symmetric about the y-axis.

An even function is a function for which replacing x with $-x$ leaves the expression that defines the function unchanged.

If a function is even, then every time the point (x, y) is on the graph, so is the point $(-x, y)$.

Even and Odd Functions

DEFINITION

An *odd function* is a function for which

$$f(-x) = -f(x) \text{ for all } x \text{ in the domain of } f$$

The graph of an odd function is symmetric about the origin.

An odd function is a function for which replacing *x* with $-x$ changes the sign of the expression that defines the function.

If a function is odd, then every time the point (x, y) is on the graph, so is the point $(-x, -y)$.

Even and Odd Functions

From the unit circle it is apparent that sine is an odd function and cosine is an even function.

We can generalize this result by drawing an angle θ and its opposite $-\theta$ in standard position and then labeling the points where their terminal sides intersect the unit circle with (x, y) and (x, $-y$), respectively. Refer Figure 19.

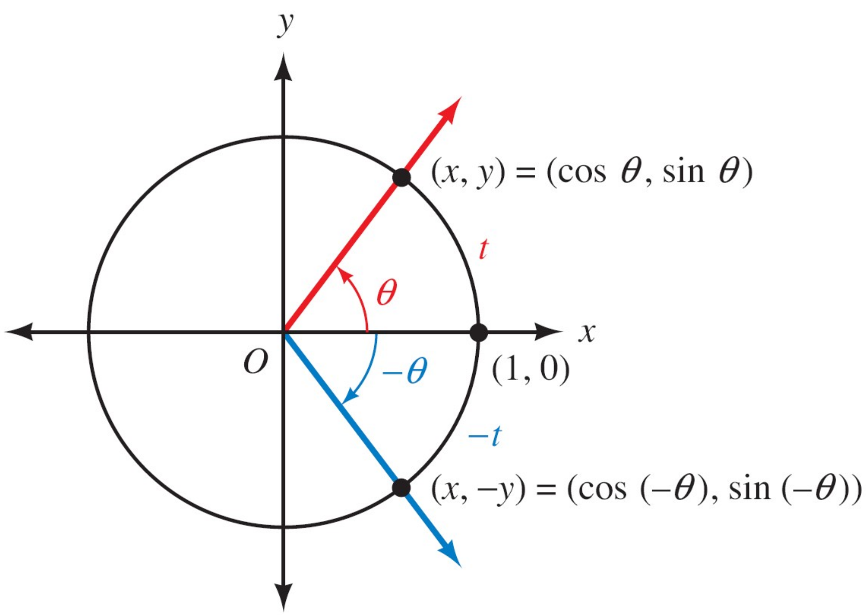


Figure 19

Even and Odd Functions

On the unit circle, $\cos\theta = x$ and $\sin\theta = y$, so we have

$$\cos(-\theta) = x = \cos\theta$$

indicating that cosine is an even function and

$$\sin(-\theta) = -y = -\sin\theta$$

indicating that sine is an odd function.

Even and Odd Functions

Now that we have established that sine is an odd function and cosine is an even function, we can use our ratio and reciprocal identities to find which of the other trigonometric functions are even and which are odd.

Example 3 shows how this is done for the cosecant function.

Example 3

Show that cosecant is an odd function.

Solution:

We must prove that $\csc(-\theta) = -\csc\theta$. That is, we must turn $\csc(-\theta)$ into $-\csc\theta$. Here is how it goes:

$$\csc(-\theta) = \frac{1}{\sin(-\theta)} \qquad \text{Reciprocal identity}$$

$$= \frac{1}{-\sin\theta} \qquad \text{Sine is an odd function}$$

$$= -\frac{1}{\sin\theta} \qquad \text{Algebra}$$

$$= -\csc\theta \qquad \text{Reciprocal identity}$$

Even and Odd Functions

Because the sine function is odd, we can see in Figure 4 that the graph of $y = \sin x$ is symmetric about the origin.

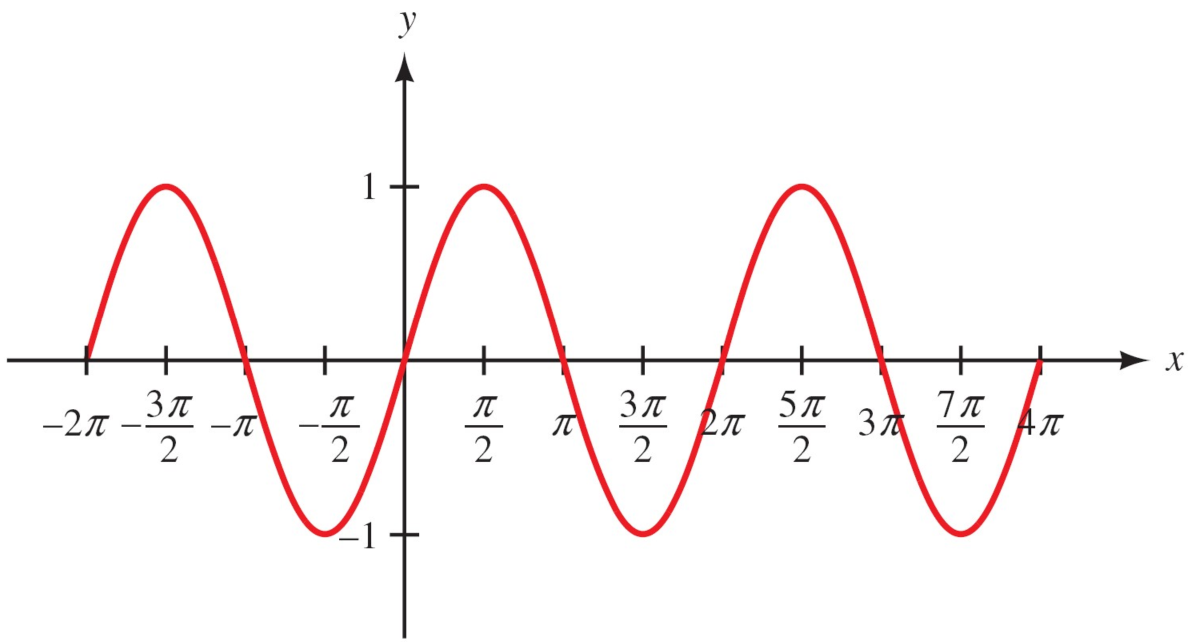


Figure 4

Even and Odd Functions

On the other hand, the cosine function is even, so the graph of $y = \cos x$ is symmetric about the y-axis as can be seen in Figure 7.

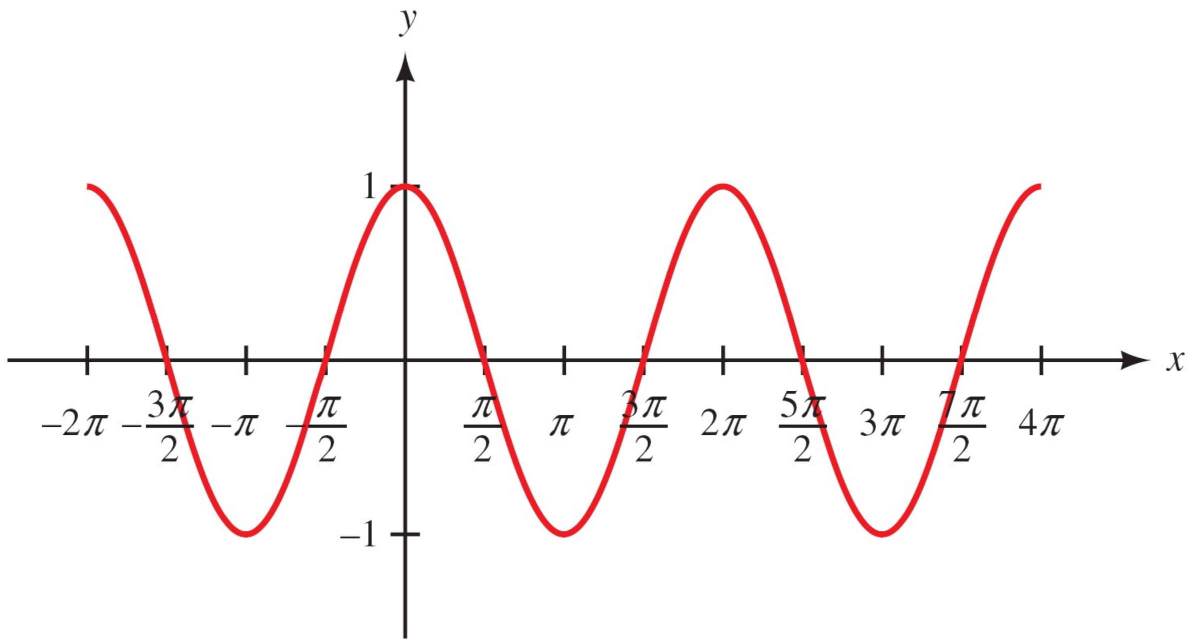


Figure 7

Even and Odd Functions

We summarize the nature of all six trigonometric functions for easy reference.

Even Functions	**Odd Functions**
$y = \cos x, \quad y = \sec x$	$y = \sin x, \quad y = \csc x$
	$y = \tan x, \quad y = \cot x$
Graphs are symmetric about the y-axis	Graphs are symmetric about the origin

Example 4

Use the even and odd function relationships to find exact values for each of the following.

a. $\cos\left(-\frac{2\pi}{3}\right)$ **b.** $\csc(-225°)$

Solution:

a. $\cos\left(-\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$ Cosine is an even function

$= -\frac{1}{2}$ Unit circle

Example 4 – *Solution*

cont'd

b. $\csc(-225°) = \dfrac{1}{\sin(-225°)}$ Reciprocal identity

$= \dfrac{1}{-\sin 225°}$ Sine is an odd function

$= \dfrac{1}{-(-1/\sqrt{2})}$ Unit circle

$= \sqrt{2}$