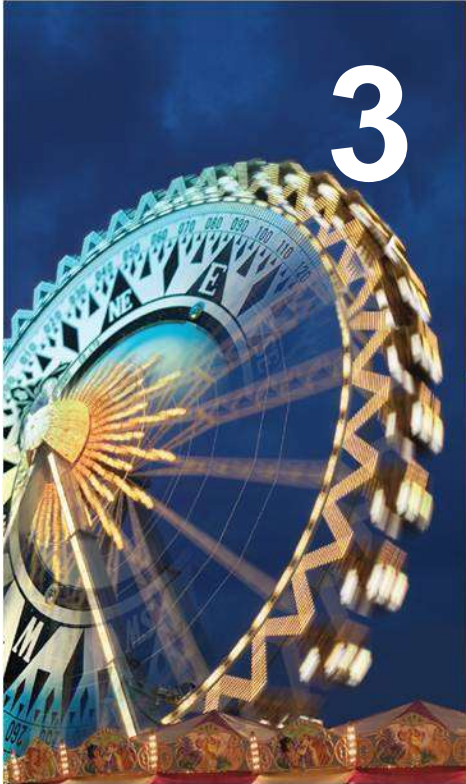




3

Radian Measure

SECTION 3.3

Definition III: Circular Functions

Learning Objectives

- 1 Evaluate a trigonometric function using the unit circle.
- 2 Find the value of a trigonometric function given a point on the unit circle.
- 3 Use a calculator to approximate the value of a trigonometric function for an angle in radians.
- 4 Use the unit circle to answer a conceptual question about a trigonometric function.

Definition III: Circular Functions

The origins of the trigonometric functions are actually found in astronomy and the need to find the length of the chord subtended by the central angle of a circle.

The Greek mathematician Hipparchus is believed to have been the first to produce a table of chords in 140 B.C., making him the founder of trigonometry in the eyes of many.

Definition III: Circular Functions

This table is essentially a table of values of the sine function, because the sine of a central angle on the unit circle is half the chord of twice the angle (Figure 1). In modern notation,

$$\text{chord}(\theta) = AC = 2AB = 2 \sin\left(\frac{\theta}{2}\right)$$

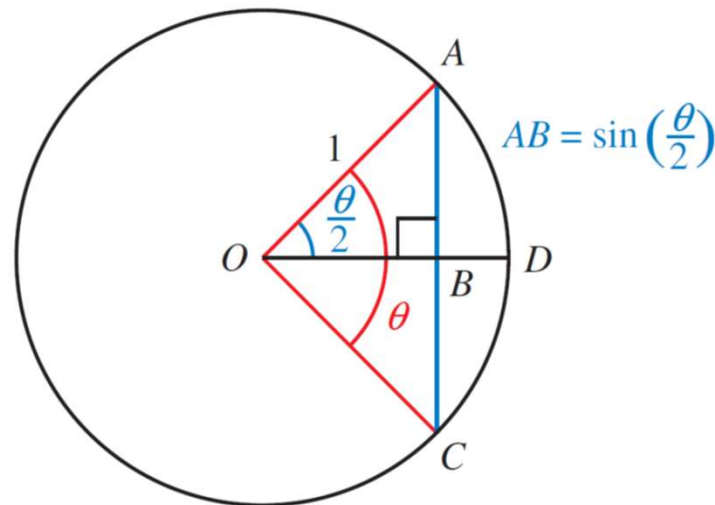


Figure 1

Definition III: Circular Functions

The unit circle (Figure 2) is the circle with center at the origin and radius 1. The equation of the unit circle is $x^2 + y^2 = 1$.

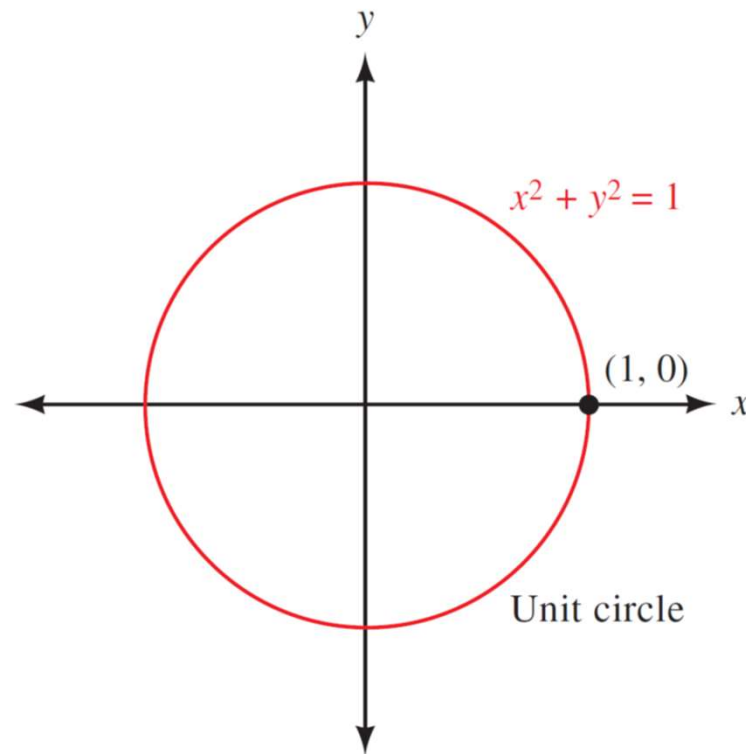


Figure 2

Definition III: Circular Functions

Suppose the terminal side of angle θ , in standard position, intersects the unit circle at point (x, y) as shown in Figure 3.

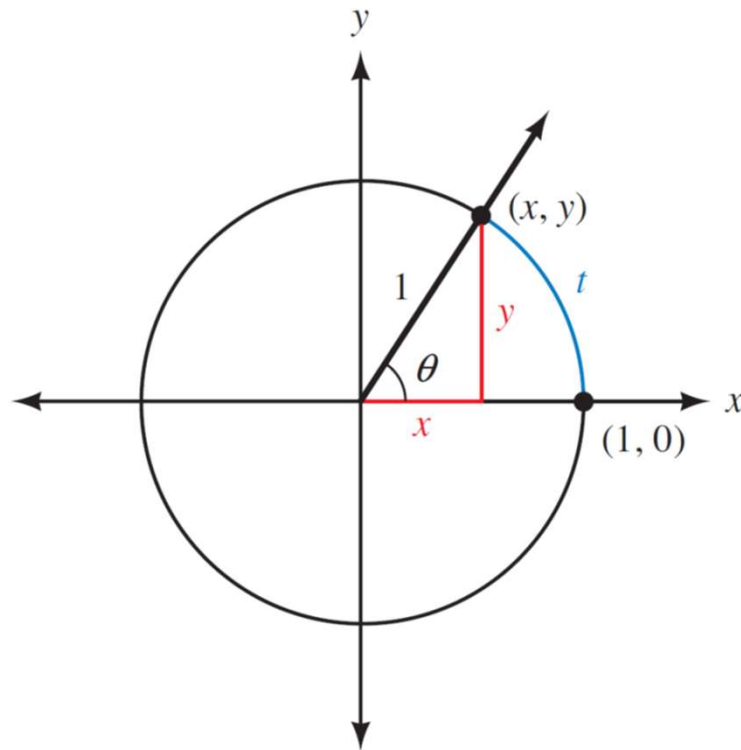


Figure 3

Definition III: Circular Functions

Because the radius of the unit circle is 1, the distance from the origin to the point (x, y) is 1.

By the first definition for the trigonometric functions we have,

$$\cos \theta = \frac{x}{r} = \frac{x}{1} = x \quad \text{and} \quad \sin \theta = \frac{y}{r} = \frac{y}{1} = y$$

The length of the arc from $(1, 0)$ to (x, y) is exactly the same as the radian measure of angle θ . Therefore, we can write

$$\cos \theta = \cos t = x \quad \text{and} \quad \sin \theta = \sin t = y$$

Definition III: Circular Functions

These results give rise to a third definition for the trigonometric functions.

DEFINITION III ■ CIRCULAR FUNCTIONS

If (x, y) is any point on the unit circle, and t is the distance from $(1, 0)$ to (x, y) along the circumference of the unit circle (Figure 4), then,

$$\cos t = x$$

$$\sin t = y$$

$$\tan t = \frac{y}{x} \quad (x \neq 0)$$

$$\cot t = \frac{x}{y} \quad (y \neq 0)$$

$$\csc t = \frac{1}{y} \quad (y \neq 0)$$

$$\sec t = \frac{1}{x} \quad (x \neq 0)$$

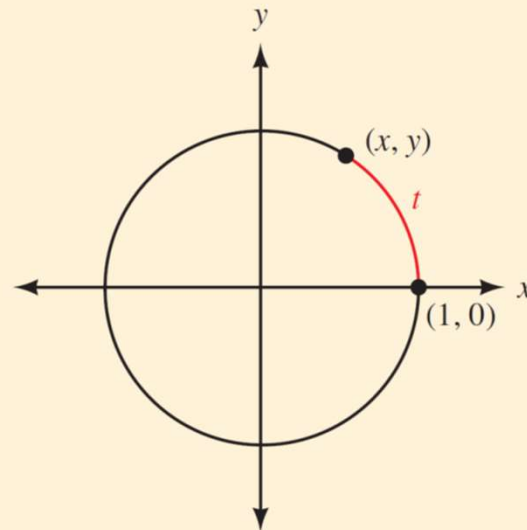


Figure 4

Definition III: Circular Functions

As we travel around the unit circle starting at $(1, 0)$, the points we come across all have coordinates $(\cos t, \sin t)$, where t is the distance we have traveled. (Note that t will be positive if we travel in the counterclockwise direction but negative if we travel in the clockwise direction.)

When we define the trigonometric functions this way, we call them *circular functions* because of their relationship to the unit circle.

Definition III: Circular Functions

Figure 5 shows an enlarged version of the unit circle with multiples of $\pi/6$ and $\pi/4$ marked off.

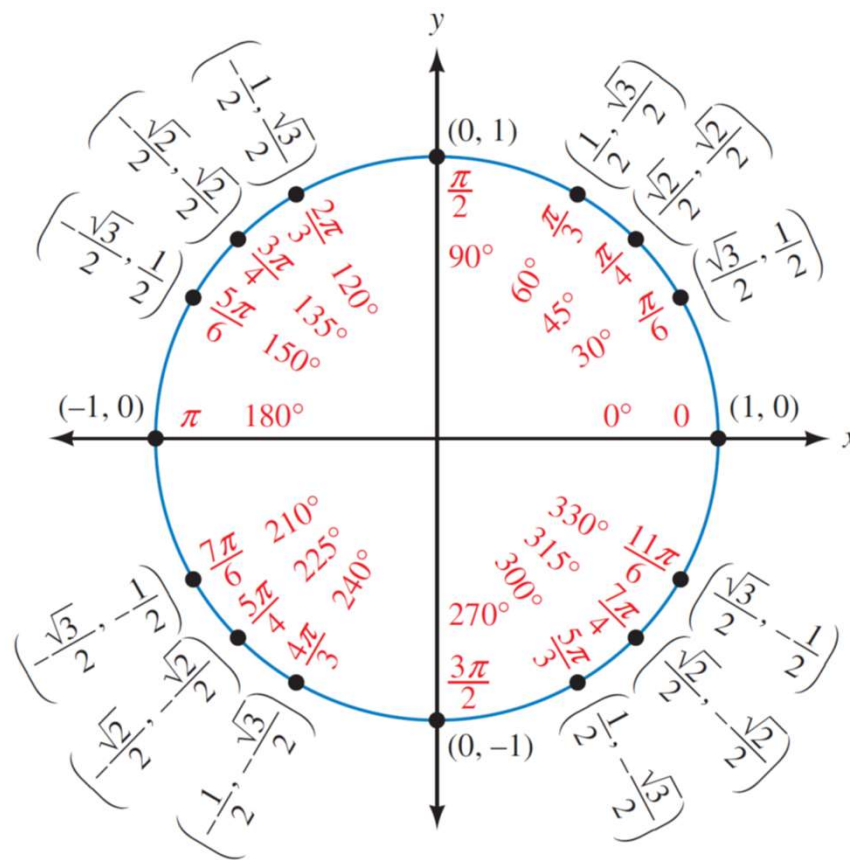


Figure 5

Definition III: Circular Functions

Each angle is given in both degrees and radians. The radian measure of each angle is the same as the distance from $(1, 0)$ to the point on the terminal side of the angle, as measured along the circumference of the circle in a counterclockwise direction.

The x - and y -coordinate of each point shown are the cosine and sine, respectively, of the associated angle or distance.

Figure 5 is helpful in visualizing the relationships among the angles shown and the trigonometric functions of those angles.

Example 1

Use Figure 5 to find the six trigonometric functions of $5\pi/6$.

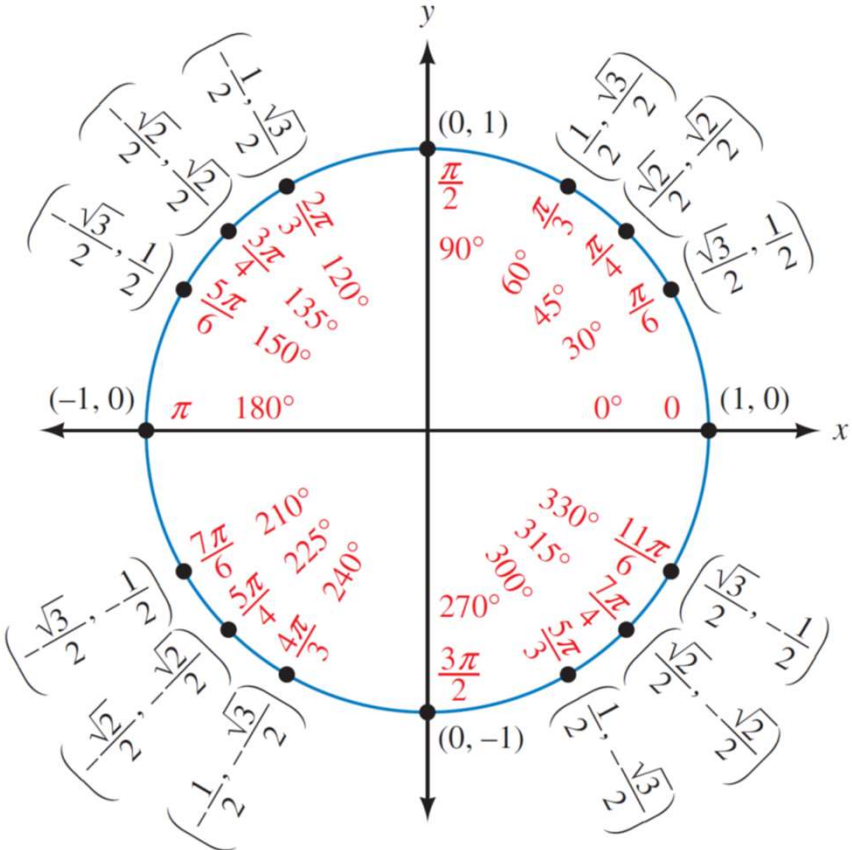


Figure 5

Example 1 – *Solution*

We obtain cosine and sine directly from Figure 5. The other trigonometric functions of $5\pi/6$ are found by using the ratio and reciprocal identities, rather than the new definition.

$$\sin \frac{5\pi}{6} = y = \frac{1}{2}$$

$$\cos \frac{5\pi}{6} = x = -\frac{\sqrt{3}}{2}$$

$$\tan \frac{5\pi}{6} = \frac{\sin (5\pi/6)}{\cos (5\pi/6)} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Example 1 – *Solution*

cont'd

$$\cot \frac{5\pi}{6} = \frac{1}{\tan (5\pi/6)} = \frac{1}{-1/\sqrt{3}} = -\sqrt{3}$$

$$\sec \frac{5\pi}{6} = \frac{1}{\cos (5\pi/6)} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\csc \frac{5\pi}{6} = \frac{1}{\sin (5\pi/6)} = \frac{1}{1/2} = 2$$

Example 3

Find $\tan t$ if t corresponds to the point $(-0.737, 0.675)$ on the unit circle (Figure 9).

Solution:

Using Definition III we have

$$\begin{aligned}\tan t &= \frac{y}{x} \\ &= \frac{0.675}{-0.737} \\ &\approx -0.916\end{aligned}$$

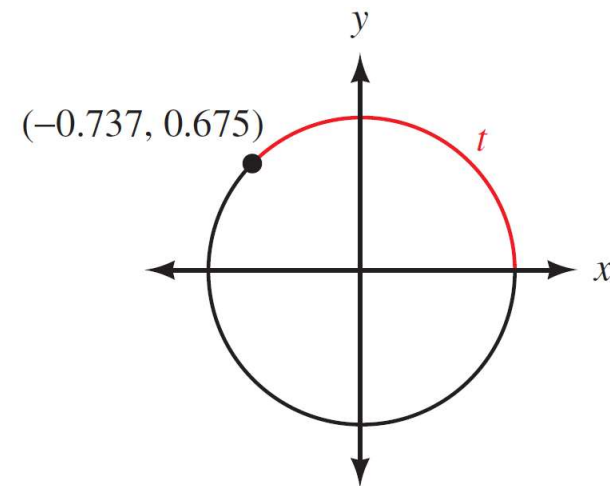


Figure 9

Definition III: Circular Functions

We have known that function is a rule that pairs each element of the domain with exactly one element from the range.

When we see the statement $y = \sin x$, it is identical to the notation $y = f(x)$.

In fact, if we wanted to be precise, we would write $y = \sin(x)$.

Definition III: Circular Functions

In visual terms, we can picture the sine function as a machine that assigns a single output value to every input value (Figure 10).

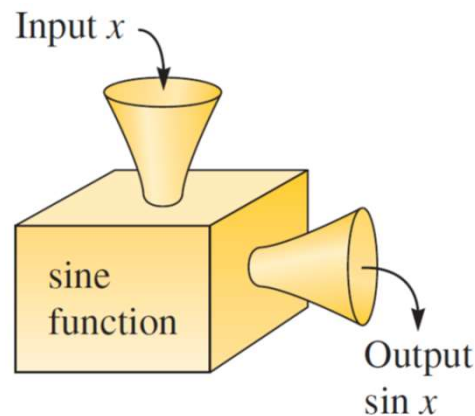


Figure 10

The input x is a real number, which can be interpreted as a distance along the circumference of the unit circle or an angle in radians. The input is formally referred to as the *argument* of the function.

Example 4

Evaluate $\sin \frac{9\pi}{4}$. Identify the function, the argument of the function, and the value of the function.

Solution:

Because

$$\frac{9\pi}{4} = \frac{\pi}{4} + \frac{8\pi}{4} = \frac{\pi}{4} + 2\pi$$

the point on the unit circle corresponding to $9\pi/4$ will be the same as the point corresponding to $\pi/4$ (Figure 11).

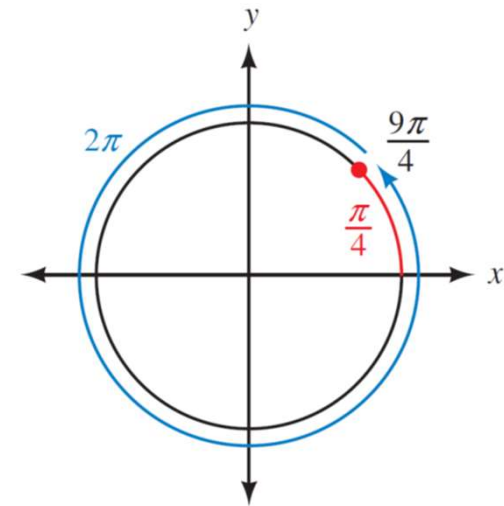


Figure 11

Example 4 – *Solution*

cont'd

Therefore,

$$\sin \frac{9\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

In terms of angles, we know this is true because $9\pi/4$ and $\pi/4$ are coterminal.

The function is the sine function, $9\pi/4$ is the argument, and $\sqrt{2}/2$ is the value of the function.



Domain and Range

Domain and Range

Using Definition III we can find the domain for each of the circular functions.

Because any value of t determines a point (x, y) on the unit circle, the sine and cosine functions are always defined and therefore have a domain of all real numbers.

Because $\tan t = y/x$ and $\sec t = 1/x$, the tangent and secant functions will be undefined when $x = 0$, which will occur at the points $(0, 1)$ and $(0, -1)$.

Domain and Range

In a similar manner, the cotangent and cosecant functions will be undefined when $y = 0$, corresponding to the points $(1, 0)$ or $(-1, 0)$.

We summarize these results here:

DOMAINS OF THE CIRCULAR FUNCTIONS

$\sin t, \cos t$	All real numbers, or $(-\infty, \infty)$
$\tan t, \sec t$	All real numbers except $t = \pi/2 + k\pi$ for any integer k
$\cot t, \csc t$	All real numbers except $t = k\pi$ for any integer k

RANGES OF THE CIRCULAR FUNCTIONS

$\sin t, \cos t$	$[-1, 1]$
$\tan t, \cot t$	All real numbers, or $(-\infty, \infty)$
$\sec t, \csc t$	$(-\infty, -1] \cup [1, \infty)$

Example 6

Determine which statements are possible for some real number z .

a. $\cos z = 2$

b. $\csc \pi = z$

c. $\tan z = 1000$

Solution:

- a.** This statement is not possible because 2 is not within the range of the cosine function. The largest value $\cos z$ can assume is 1.
- b.** This statement is also not possible, because $\csc \pi$ is undefined and therefore not equal to any real number.

Example 6 – *Solution*

cont'd

- c. This statement is possible because the range of the tangent function is all real numbers, which certainly includes 1,000.



Geometric Representations

Geometric Representations

Based on the circular definitions, we can represent the values of the six trigonometric functions geometrically as indicated in Figure 12.

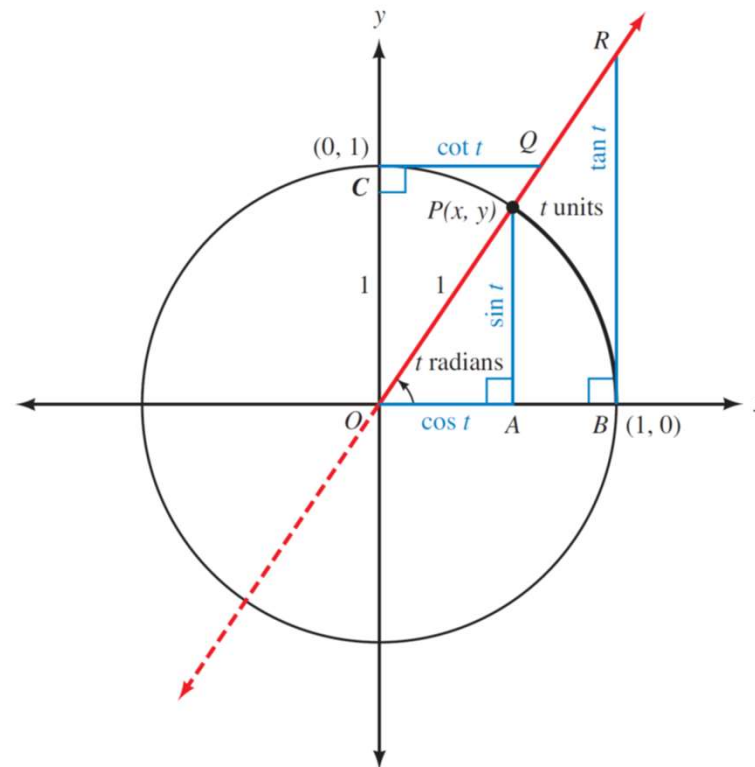


Figure 12

Geometric Representations

The diagram shows a point $P(x, y)$ that is t units from the point $(1, 0)$ on the circumference of the unit circle.

Therefore, $\cos t = x$ and $\sin t = y$. Because triangle BOR is similar to triangle AOP , we have

$$\frac{BR}{OB} = \frac{AP}{OA} = \frac{\sin t}{\cos t} = \tan t$$

Because $OB = 1$, BR is equal to $\tan t$. Notice that this will be the slope of OP . Using a similar argument, it can be shown that $CQ = \cot t$, $OQ = \csc t$, and $OR = \sec t$.

Example 7

Describe how $\sec t$ varies as t increases from 0 to $\pi/2$.

Solution:

When $t = 0$, $OR = 1$ so that $\sec t$ will begin at a value of 1.

As t increases, $\sec t$ grows larger and larger. Eventually, when $t = \pi/2$, OP will be vertical so $\sec t = OR$ will no longer be defined.