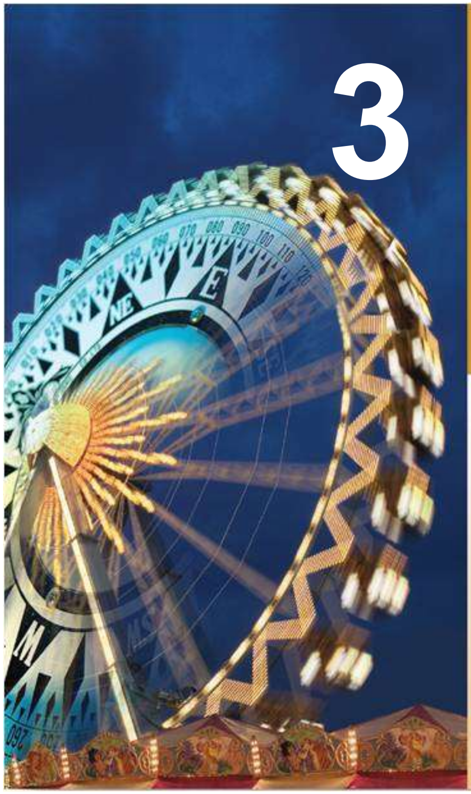


3 Radian Measure

SECTION 3.2

Radians and Degrees

Learning Objectives

1. Find the radian measure of a central angle given the radius and arc length.
2. Convert an angle from degrees to radians or vice versa.
3. Evaluate a trigonometric function using radians.
4. Identify the reference angle for a given angle measured in radians.

Radians and Degrees

The trigonometric functions we have worked with so far have had the form $y = f(\theta)$, where θ is measured in degrees.

To apply the knowledge we have about functions from algebra to our trigonometric functions, we need to write our angles as real numbers, not degrees. The key to doing this is called *radian measure*.

Radian measure is a relatively new concept in the history of mathematics.

Radians and Degrees

We have known that a central angle in a circle is an angle with its vertex at the center of the circle. Here is the definition for an angle with a measure of 1 radian.

DEFINITION

In a circle, a central angle that cuts off an arc equal in length to the radius of the circle has a measure of 1 *radian* (rad). Figure 1 illustrates this.

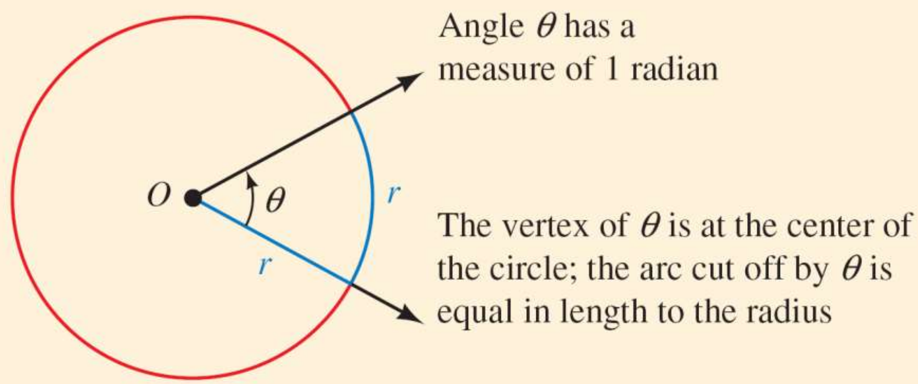


Figure 1

Radians and Degrees

To find the radian measure of *any* central angle, we must find how many radii are in the arc it cuts off. To do so, we divide the arc length by the radius.

If the radius is 2 centimeters and the arc cut off by central angle θ is 6 centimeters, then the radian measure of θ is $\frac{6}{2} = 3$ rad.

Radians and Degrees

Here is the formal definition:

DEFINITION ■ RADIAN MEASURE

If a central angle θ, in a circle of radius r, cuts off an arc of length s, then the measure of θ, in radians, is given by s/r (Figure 2).

$$\theta \text{ (in radians)} = \frac{s}{r}$$

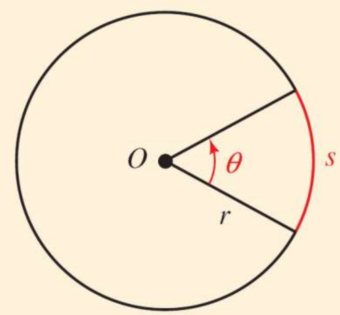


Figure 2

As you will see later in this section, one radian is equal to approximately 57.3°.

Example 1

A central angle θ in a circle of radius 3 centimeters cuts off an arc of length 6 centimeters. What is the radian measure of θ?

Solution:

We have r = 3 cm and s = 6 cm (Figure 3);

therefore,

$$\theta \text{ (in radians)} = \frac{s}{r} = \frac{6 \text{ cm}}{3 \text{ cm}} = 2$$

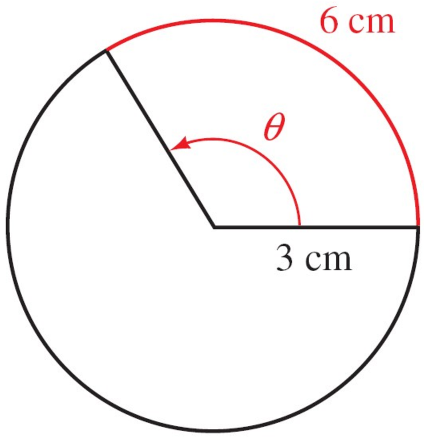


Figure 3

We say the radian measure of θ is 2, or θ = 2 rad.

Radians and Degrees

To see the relationship between degrees and radians, we can compare the number of degrees and the number of radians in one full rotation (Figure 4).

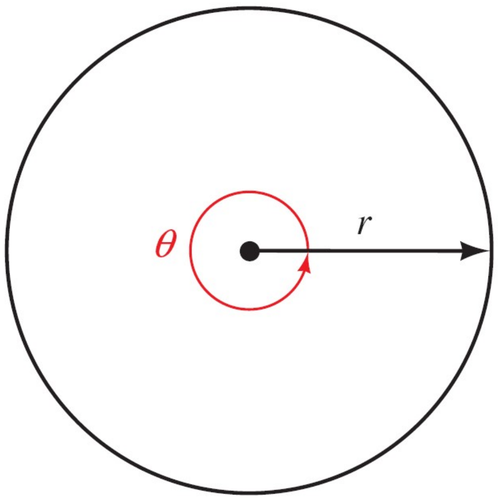


Figure 4

The angle formed by one full rotation about the center of a circle of radius *r* will cut off an arc equal to the circumference of the circle.

Radians and Degrees

Since the circumference of a circle of radius r is $2\pi r$, we have

θ measures one full rotation $\qquad \theta = \dfrac{2\pi r}{r} = 2\pi \qquad$ The measure of θ in radians is 2π

Because one full rotation in degrees is 360°, we have the following relationship between radians and degrees.

$$360° = 2\pi \text{ rad}$$

Dividing both sides by 2 we have

$$180° = \pi \text{ rad}$$

Radians and Degrees

To obtain conversion factors that will allow us to change back and forth between degrees and radians, we divide both sides of this last equation alternately by 180 and by π.

$$180° = \pi \text{ rad}$$

Divide both sides by 180

$$1° = \frac{\pi}{180} \text{ rad}$$

Divide both sides by π

$$\left(\frac{180}{\pi}\right)^{\circ} = 1 \text{ rad}$$

To gain some insight into the relationship between degrees and radians, we can approximate π with 3.14 to obtain the approximate number of degrees in 1 radian.

$$1 \text{ rad} = 1\left(\frac{180}{\pi}\right)^{\circ} \approx 1\left(\frac{180}{3.14}\right)^{\circ} = 57.3°$$

To the nearest tenth

Converting from Degrees to Radians

Example 2

Convert 45° to radians.

Solution:

Because 1° = $\frac{\pi}{180}$ radians, and 45° is the same as 45(1°),

we have

$$45^\circ = 45\left(\frac{\pi}{180}\right)\text{ rad} = \frac{\pi}{4}\text{ rad}$$

as illustrated in Figure 6.

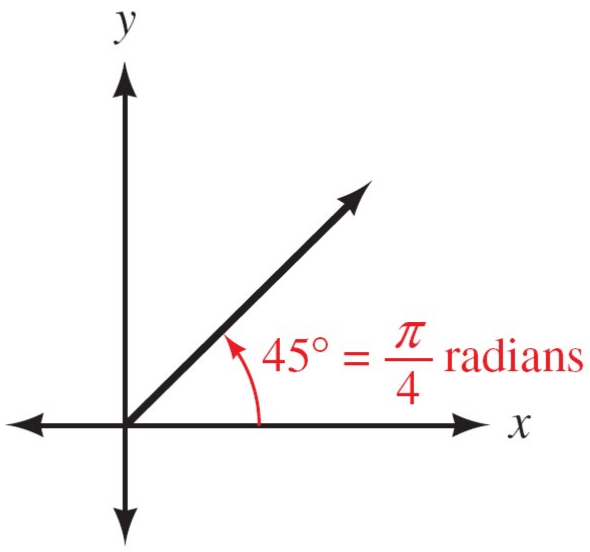


Figure 6

Example 2 – *Solution*

cont'd

When we have our answer in terms of π, as in $\pi/4$, we are writing an exact value.

If we wanted a decimal approximation, we would substitute 3.14 for π.

$$\text{Exact value} \qquad \frac{\pi}{4} \approx \frac{3.14}{4} = 0.785 \qquad \text{Approximate value}$$

Converting from Radians to Degrees

Example 5

Convert $\pi/6$ to degrees.

Solution:

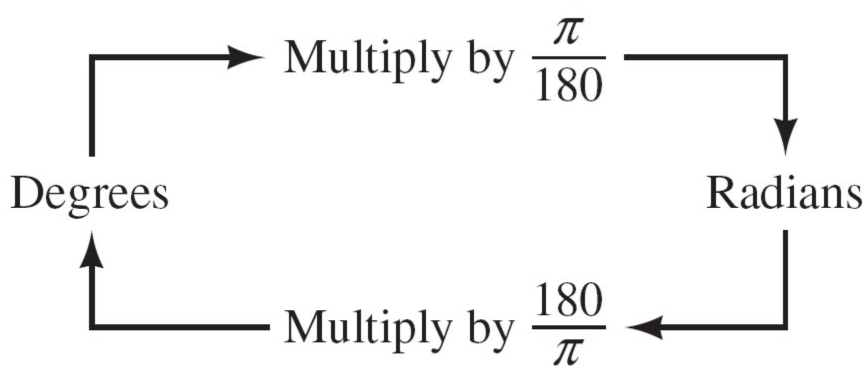
To convert from radians to degrees, we multiply by $180/\pi$.

$$\frac{\pi}{6} \text{ (rad)} = \frac{\pi}{6}\left(\frac{180}{\pi}\right)^{\circ}$$

$$= 30^{\circ}$$

Converting from Radians to Degrees

As is apparent from the preceding examples, changing from degrees to radians and radians to degrees is simply a matter of multiplying by the appropriate conversion factors.



Converting from Radians to Degrees

Table 1 displays the conversions between degrees and radians for the special angles and also summarizes the exact values of the sine, cosine, and tangent of these angles for your convenience.

θ		Value of Trigonometric Function		
Degrees	*Radians*	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	undefined
$180°$	π	0	-1	0
$270°$	$\frac{3\pi}{2}$	-1	0	undefined
$360°$	2π	0	1	0

Table 1

Converting from Radians to Degrees

In Figure 12 we show each of the special angles in both degrees and radians for all four quadrants.

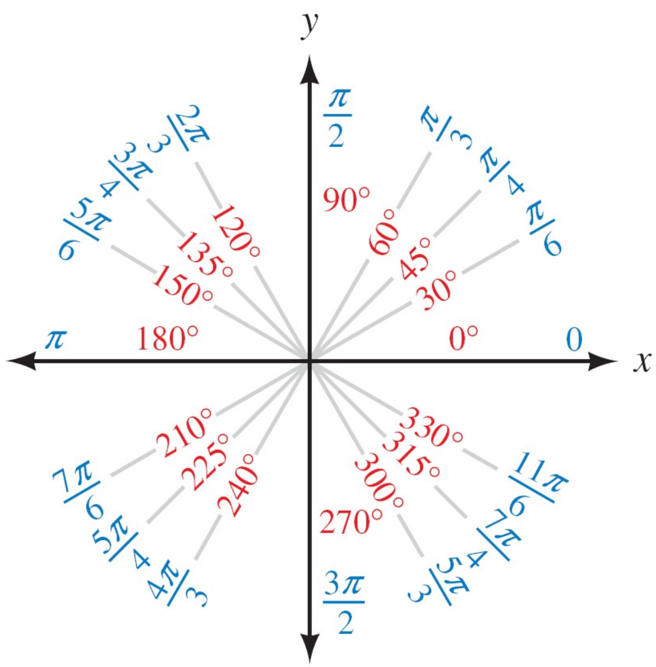


Figure 12

Example 8

Find sin $\frac{\pi}{6}$.

Solution:

Because $\pi/6$ and 30° are equivalent, so are their sines.

$$\sin \frac{\pi}{6} = \sin 30^\circ$$

$$= \frac{1}{2}$$

Example 9

Find $4 \sin \frac{7\pi}{6}$.

Solution:

Because $7\pi/6$ is just slightly larger than $\pi = 6\pi/6$, we know that $7\pi/6$ terminates in QIII and therefore its sine will be negative (Figure 13).

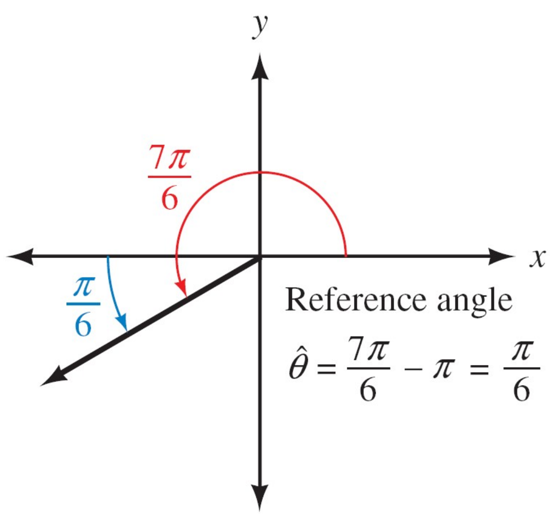


Figure 13

Example 9 – *Solution*

cont'd

The reference angle is

$$\frac{7\pi}{6} - \pi = \frac{7\pi}{6} - \frac{6\pi}{6} = \frac{\pi}{6}$$

Then

$$4 \sin \frac{7\pi}{6} = 4\left(-\sin \frac{\pi}{6}\right) = 4\left(-\frac{1}{2}\right) = -2$$