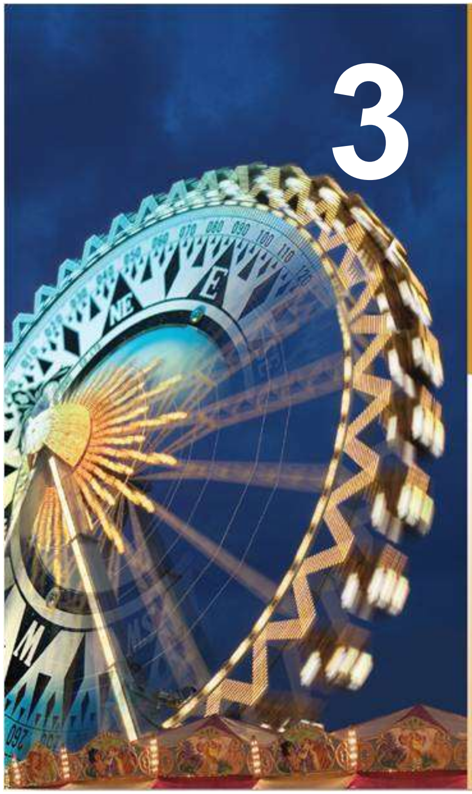


3 Radian Measure

SECTION 3.1

Reference Angle

Learning Objectives

1. Identify the reference angle for a given angle in standard position.
2. Use a reference angle to find the exact value of a trigonometric function.
3. Use a calculator to approximate the value of a trigonometric function.
4. Find an angle given the quadrant and the value of a trigonometric function.

Reference Angle

We found exact values for trigonometric functions of certain angles between 0° and 90° in earlier chapter. By using what are called *reference angles*, we can find exact values for trigonometric functions of angles outside the interval 0° to 90°.

DEFINITION

The *reference angle* (sometimes called *related angle*) for any angle θ in standard position is the positive acute angle between the terminal side of θ and the x-axis. In this book, we will denote the reference angle for θ by $\hat{\theta}$.

Note that, for this definition, $\hat{\theta}$ is always positive and always between 0° and 90°. That is, a reference angle is always an acute angle.

Example 1

Name the reference angle for each of the following angles.

a. 30° **b.** 135°

c. 240° **d.** 330°

e. –210° **f.** –140°

Example 1 – *Solution*

We draw each angle in standard position. The reference angle is the positive acute angle formed by the terminal side of the angle in question and the *x*-axis (Figure 1).

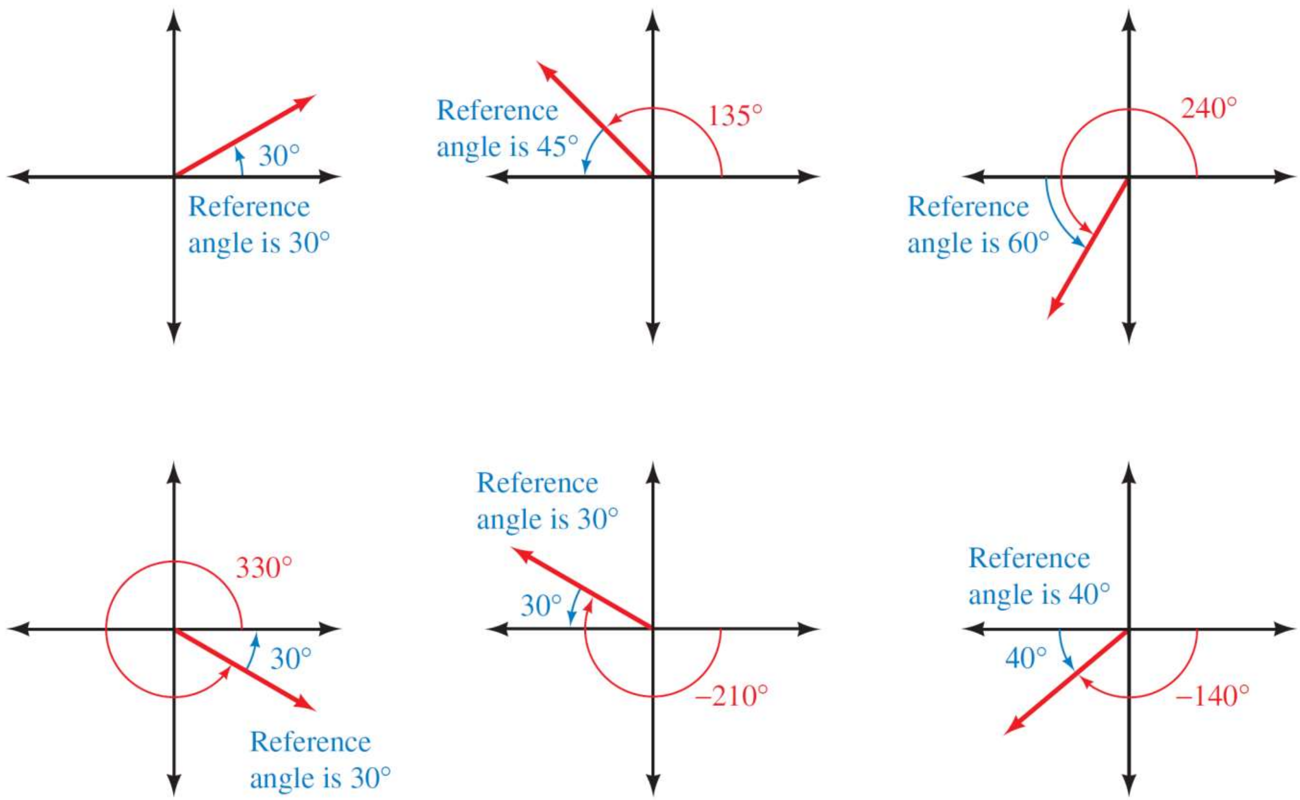


Figure 1

Reference Angle

We can generalize the results of Example 1 as follows: If θ is a positive angle between 0° and 360°, and

$$\text{if } \theta \in \text{QI}, \qquad \text{then } \hat{\theta} = \theta$$

$$\text{if } \theta \in \text{QII}, \qquad \text{then } \hat{\theta} = 180^\circ - \theta$$

$$\text{if } \theta \in \text{QIII}, \qquad \text{then } \hat{\theta} = \theta - 180^\circ$$

$$\text{if } \theta \in \text{QIV}, \qquad \text{then } \hat{\theta} = 360^\circ - \theta$$

Reference Angle

We can use our information on reference angles and the signs of the trigonometric functions to write the following theorem.

REFERENCE ANGLE THEOREM

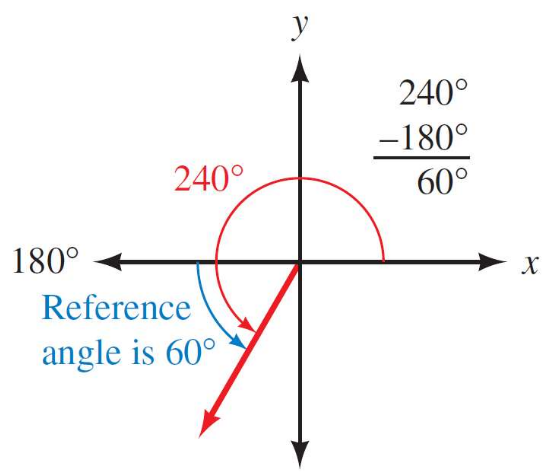
A trigonometric function of an angle and its reference angle are the same, except, perhaps, for a difference in sign.

Example 2

Find the exact value of sin 240°.

Solution:

Figure 3 is a diagram of the situation.



Find the exact value of sin 210°.

Figure 3

Example 2 – *Solution*

cont'd

Step 1 We find $\hat{\theta}$ by subtracting 180° from θ.

$$\hat{\theta} = 240^\circ - 180^\circ = 60^\circ$$

Step 2 Since θ terminates in quadrant III, and the sine function is negative in quadrant III, our answer will be negative. That is, $\sin \theta = -\sin \hat{\theta}$.

Step 3 Using the results of Steps 1 and 2, we write

$$\sin 240^\circ = -\sin 60^\circ$$

Example 2 – *Solution*

cont'd

Step 4 We finish by finding sin 60°.

$$\sin 240^\circ = -\sin 60^\circ \qquad \text{Sine is negative in QIII}$$

$$= -\left(\frac{\sqrt{3}}{2}\right) \qquad \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$= -\frac{\sqrt{3}}{2}$$

Reference Angle

The trigonometric functions of an angle and any angle coterminal to it are always equal.

For sine and cosine, we can write this in symbols as follows:

for any integer *k*,

$$\sin(\theta + 360°k) = \sin\theta \qquad \text{and} \qquad \cos(\theta + 360°k) = \cos\theta$$

To find values of trigonometric functions for an angle larger than 360° or smaller than 0°, we simply find an angle between 0° and 360° that is coterminal to it and then use the steps outlined in Example 2.

Example 5

Find the exact value of cos 495°.

Solution:

By subtracting 360° from 495°, we obtain 135°, which is coterminal to 495°.

The reference angle for 135° is 45°.

Because 495° terminates in quadrant II, its cosine is negative (Figure 7).

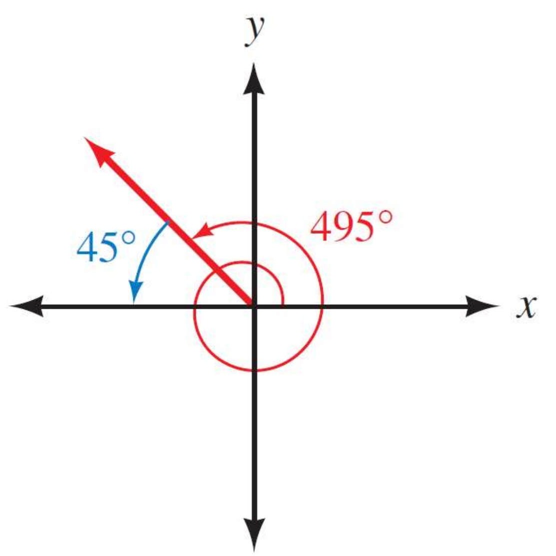


Figure 7

Example 5 – *Solution*

cont'd

$$\cos 495° = \cos 135° \qquad \text{495° and 135° are coterminal}$$

$$= -\cos 45° \qquad \text{In QII } \cos\theta = -\cos\hat{\theta}$$

$$= -\frac{\sqrt{2}}{2} \qquad \text{Exact value}$$

Approximations

Approximations

To find trigonometric functions of angles that do not lend themselves to exact values, we use a calculator. To find an approximation for sin θ, cos θ, or tan θ, we simply enter the angle and press the appropriate key on the calculator.

Check to see that you can obtain the following values for sine, cosine, and tangent of 250° and –160° on your calculator. (These answers are rounded to the nearest ten-thousandth.)

Approximations

Make sure your calculator is set to degree mode.

$$\sin 250^\circ = -0.9397 \qquad \sin(-160^\circ) = -0.3420$$

$$\cos 250^\circ = -0.3420 \qquad \cos(-160^\circ) = -0.9397$$

$$\tan 250^\circ = 2.7475 \qquad \tan(-160^\circ) = 0.3640$$

Example 6

Find θ to the nearest degree if $\sin\theta = -0.5592$ and θ terminates in QIII with $0° \le \theta < 360°$.

Solution:

First we find the reference angle using the $\boxed{\sin^{-1}}$ key with the positive value 0.5592. From this, we get $\hat{\theta} = 34°$.

As shown in Figure 8, the desired angle in QIII whose reference angle is 34° is

$$\begin{aligned}\theta &= 180° + \hat{\theta}\\ &= 180° + 34°\\ &= 214°\end{aligned}$$

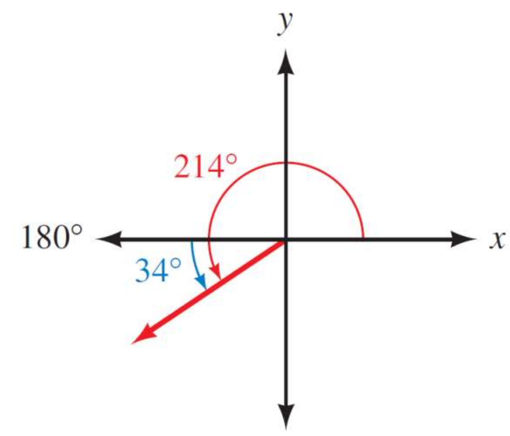


Figure 8

Example 6 – *Solution*

cont'd

If we wanted to list *all* the angles that terminate in QIII and have a sine of –0.5592, we would write

$$\theta = 214° + 360°k \qquad \text{where } k \text{ is any integer}$$

This gives us all angles coterminal with 214°.

Example 10

Find θ to the nearest degree if cot θ = –1.6003 and θ terminates in QII, with $0° \leq \theta < 360°$.

Solution:

To find the reference angle on a calculator, we ignore the negative sign in –1.6003 and use the fact that cot θ is the reciprocal of tan θ.

$$\text{If } \cot \theta = 1.6003, \quad \text{then } \tan \theta = \frac{1}{1.6003}$$

Example 10 – *Solution*

cont'd

From this last line we see that the keys to press are

Scientific Calculator	Graphing Calculator
1.6003 $\boxed{1/x}$ $\boxed{\tan^{-1}}$	$\boxed{\tan^{-1}}$ $\boxed{(}$ 1.6003 $\boxed{x^{-1}}$ $\boxed{)}$ $\boxed{\text{ENTER}}$

To the nearest degree, the reference angle is $\hat{\theta}$ = 32°. Because we want θ to terminate in QII, we subtract 32° from 180° to get θ = 148°.

Example 10 – *Solution*

cont'd

Again, we can check our result on a calculator by entering 148°, finding its tangent, and then finding the reciprocal of the result.

Scientific Calculator	**Graphing Calculator**
148 $\boxed{\tan}$ $\boxed{1/x}$	1 $\boxed{\div}$ $\boxed{\tan}$ $\boxed{(}$ 148 $\boxed{)}$ $\boxed{\text{ENTER}}$

The calculator gives a result of approximately –1.6003.