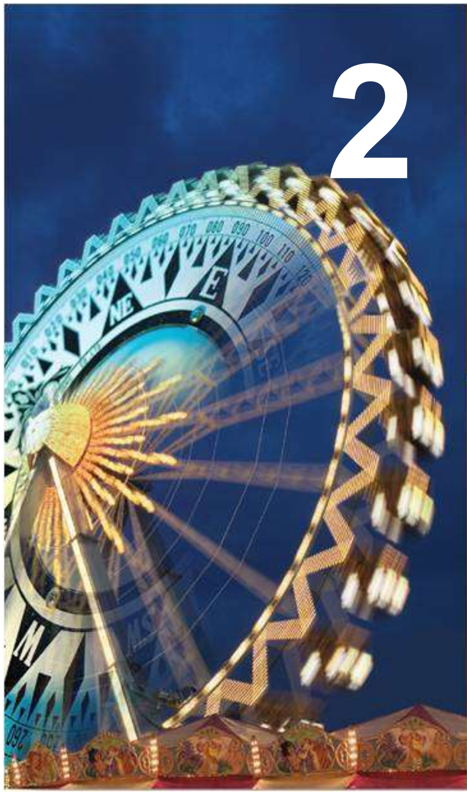

2 Right Triangle Trigonometry

SECTION 2.5

Vectors: A Geometric Approach

Learning Objectives

1. Find the magnitude of the horizontal and vertical vector components for a vector.
2. Find the magnitude of a vector and the angle it makes with the positive *x*-axis.
3. Solve an applied problem using vectors.
4. Compute the work done by a force with a given distance.

Vectors: A Geometric Approach

Many of the quantities that describe the world around us have both magnitude and direction, while others have only magnitude.

Quantities that have magnitude and direction are called *vector quantities*, while quantities with magnitude only are called *scalars*.

One way to represent vector quantities geometrically is with arrows. The direction of the arrow represents the direction of the vector quantity, and the length of the arrow corresponds to the magnitude.

Vectors: A Geometric Approach

NOTATION

To distinguish between vectors and scalars, we will write the letters used to represent vectors with boldface type, such as **U** or **V**. (When you write them on paper, put an arrow above them like this: $\overrightarrow{\mathbf{U}}$ or $\overrightarrow{\mathbf{V}}$.) The magnitude of a vector is represented with absolute value symbols. For example, the magnitude of **V** is written $|\mathbf{V}|$. Table 1 illustrates further.

Notation	The quantity is
$\mathbf{V}$	a vector
$\overrightarrow{\mathbf{V}}$	a vector
$\overrightarrow{\mathbf{AB}}$	a vector
x	a scalar
$\|\mathbf{V}\|$	the magnitude of vector **V**, a scalar

Table 1

Zero Vector

Zero Vector

A vector having a magnitude of zero is called a *zero vector* and is denoted by **0**. A zero vector has no defined direction.

Equality for Vectors

Equality for Vectors

The position of a vector in space is unimportant. Two vectors are equivalent if they have the same magnitude and direction.

In Figure 2, $\mathbf{V}_1 = \mathbf{V}_2 \neq \mathbf{V}_3$. The vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are equivalent because they have the same magnitude and the same direction.

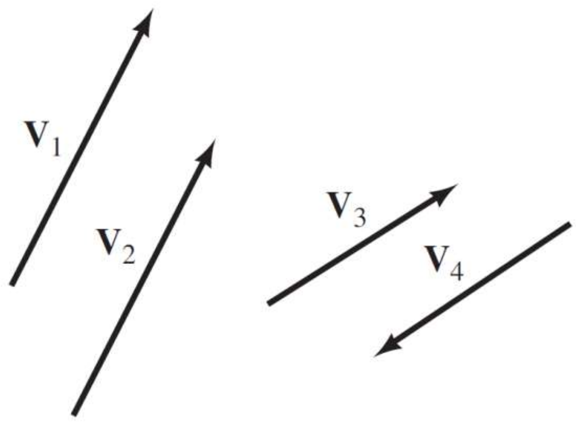

Figure 2

Equality for Vectors

Notice also that $\mathbf{V}_3$ and $\mathbf{V}_4$ have the same magnitude but opposite directions. This means that $\mathbf{V}_4$ is the opposite of $\mathbf{V}_3$, or $\mathbf{V}_4 = -\mathbf{V}_3$.

Addition and Subtraction of Vectors

Addition and Subtraction of Vectors

The sum of the vectors **U** and **V**, written **U** + **V**, is called the *resultant vector*.

It is the vector that extends from the tail of **U** to the tip of **V** when the tail of **V** is placed at the tip of **U**, as illustrated in Figure 3.

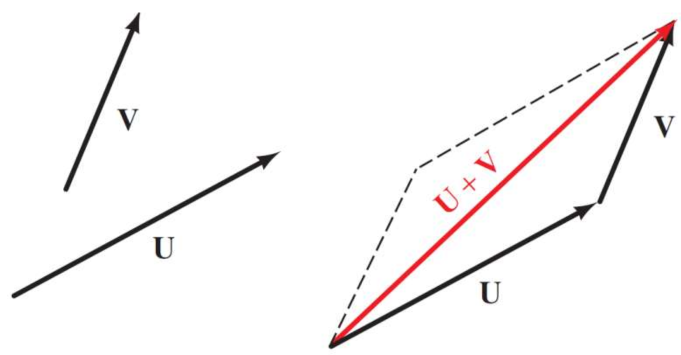

Figure 3

Note that this diagram shows the resultant vector to be a diagonal in the parallelogram that has **U** and **V** as adjacent sides.

Addition and Subtraction of Vectors

This being the case, we could also add the vectors by putting the tails of **U** and **V** together to form adjacent sides of that same parallelogram, as shown in Figure 4.

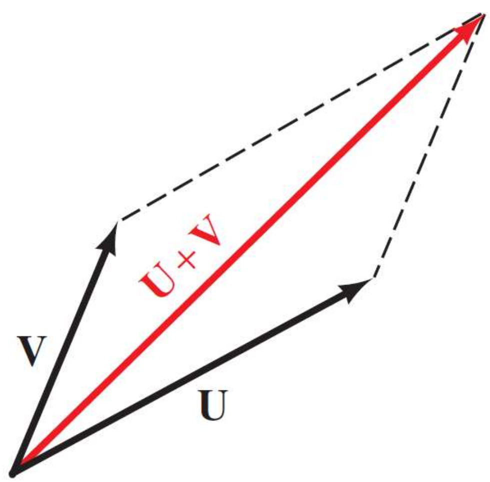

Figure 4

In either case, the resultant vector is the diagonal that starts at the tail of **U**.

Addition and Subtraction of Vectors

To subtract one vector from another, we can add its opposite. That is,

$$\mathbf{U} - \mathbf{V} = \mathbf{U} + (-\mathbf{V})$$

If **U** and **V** are the vectors shown in Figure 3, then their difference, **U** – **V**, is shown in Figure 5.

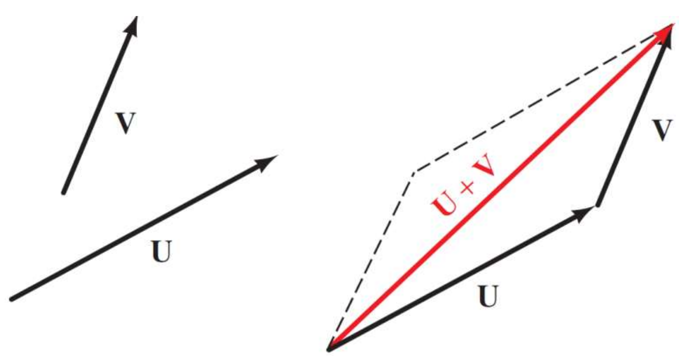

Figure 3

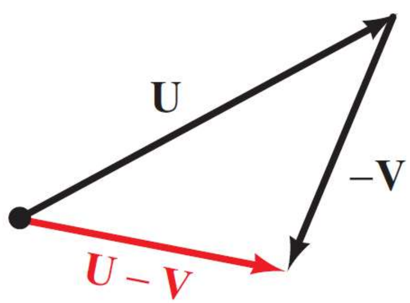

Figure 5

Addition and Subtraction of Vectors

Another way to find **U** – **V** is to put the tails of **U** and **V** together and then draw a vector from the tip of **V** to the tip of **U**, completing a triangle as shown in Figure 6.

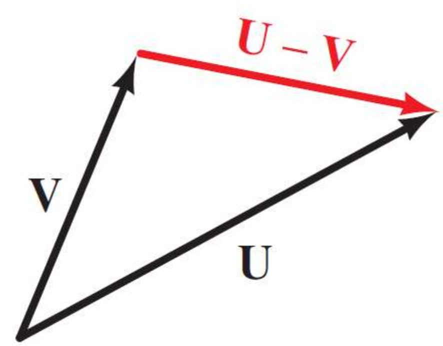

Figure 6

Example 1

A boat is crossing a river that runs due north. The boat is pointed due east and is moving through the water at 12 miles per hour. If the current of the river is a constant 5.1 miles per hour, find the actual course of the boat through the water to two significant digits.

Solution:

Problems like this are a little difficult to read the first time they are encountered.

Example 1 – *Solution*

cont'd

Even though the boat is "headed" due east, as it travels through the water the water itself is moving northward, so it is actually on a course that will take it east and a little north.

It may help to imagine the first hour of the journey as a sequence of separate events. First, the boat travels 12 miles directly across the river while the river remains still.

We can represent this part of the trip by a vector pointing due east with magnitude 12. Then the boat turns off its engine, and we allow the river to flow while the boat remains still (although the boat will be carried by the river).

Example 1 – *Solution*

cont'd

We use a vector pointing due north with magnitude 5.1 to represent this second part of the journey. The end result is that each hour the boat travels both east and north (Figure 7).

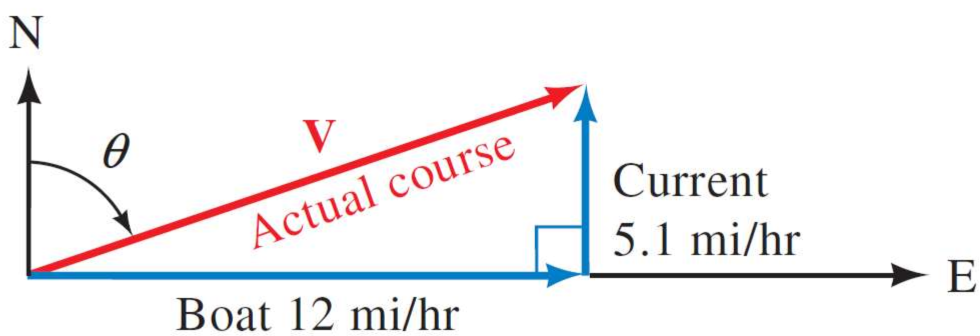

Figure 7

Example 1 – *Solution*

cont'd

We find θ using a tangent ratio. Note that the angle between the vector representing the actual course and the current vector is also θ.

$$\tan \theta = \frac{12}{5.1}$$

$$\tan \theta = 2.3529$$

$$\theta = \tan^{-1}(2.3529)$$

$$= 67^\circ \quad \text{To the nearest degree}$$

Example 1 – *Solution*

cont'd

If we let **V** represent the actual course of the boat, then we can find the magnitude of **V** using the Pythagorean Theorem or a trigonometric ratio.

Using the Pythagorean Theorem, we have

$$|\mathbf{V}| = \sqrt{12^2 + 5.1^2}$$

$$= 13 \quad \text{To two significant digits}$$

The actual course of the boat is 13 miles per hour at N 67° E. That is, the vector **V**, which represents the motion of the boat with respect to the banks of the river, has a magnitude of 13 miles per hour and a direction of N 67° E.

Horizontal and Vertical Vector Components

Horizontal and Vertical Vector Components

Two horizontal and vertical vectors whose sum is **V** are shown in Figure 9. Note that in Figure 9 we labeled the horizontal vector as $\mathbf{V}_x$ and the vertical as $\mathbf{V}_y$.

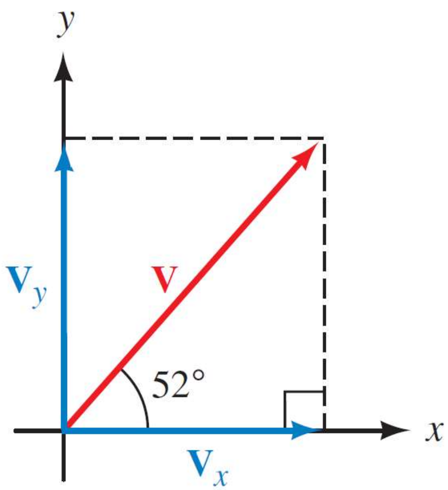

Figure 9

We call $\mathbf{V}_x$ the *horizontal vector component* of **V** and $\mathbf{V}_y$ the *vertical vector component* of **V**.

Horizontal and Vertical Vector Components

This leads us to the following general result.

VECTOR COMPONENTS

If $\mathbf{V}$ is a vector in standard position and θ is the angle measured from the positive x-axis to $\mathbf{V}$, then the horizontal and vertical vector components of $\mathbf{V}$ are given by

$$\mathbf{V}_x = |\mathbf{V}| \cos \theta \quad \text{and} \quad \mathbf{V}_y = |\mathbf{V}| \sin \theta$$

Example 2

The human cannonball is shot from a cannon with an initial velocity of 53 miles per hour at an angle of 60° from the horizontal. Find the magnitudes of the horizontal and vertical vector components of the velocity vector.

Solution:

Figure 10 is a diagram of the situation.

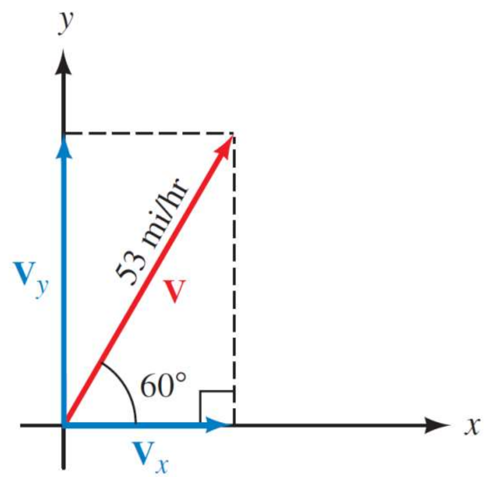

Figure 10

Example 2 – *Solution*

cont'd

The magnitudes of $\mathbf{V}_x$ and $\mathbf{V}_y$ from Figure 10 to two significant digits are as follows:

$$|\mathbf{V}_x| = 53 \cos 60°$$

$$= 27 \text{ mi/hr}$$

$$|\mathbf{V}_y| = 53 \sin 60°$$

$$= 46 \text{ mi/hr}$$

The human cannonball has a horizontal velocity of 27 miles per hour and an initial vertical velocity of 46 miles per hour.

Horizontal and Vertical Vector Components

The magnitude of a vector can be written in terms of the magnitude of its horizontal and vertical vector components (Figure 11).

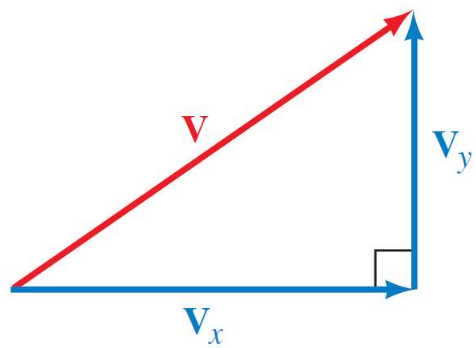

Figure 11

By the Pythagorean Theorem we have

$$|\mathbf{V}| = \sqrt{|\mathbf{V}_x|^2 + |\mathbf{V}_y|^2}$$

Example 3

An arrow is shot into the air so that its horizontal velocity is 25 feet per second and its vertical velocity is 15 feet per second. Find the velocity of the arrow.

Solution:

Figure 12 shows the velocity vector along with the angle of elevation of the velocity vector.

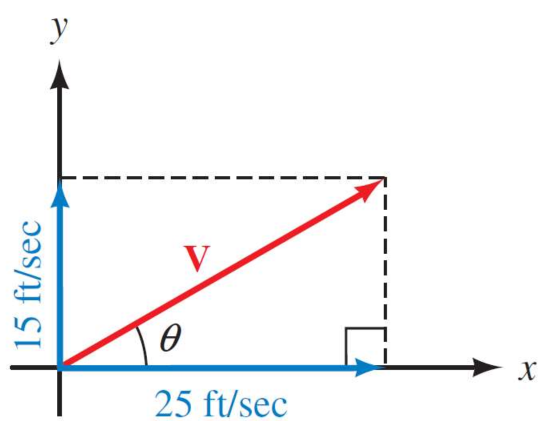

Figure 12

Example 3 – *Solution*

cont'd

The magnitude of the velocity is given by

$$|\mathbf{V}| = \sqrt{25^2 + 15^2}$$

$$= 29 \text{ ft/sec}$$ To the nearest whole number

We can find the angle of elevation using a tangent ratio.

$$\tan\theta = \frac{|\mathbf{V}_y|}{|\mathbf{V}_x|} = \frac{15}{25} = 0.6$$

$$\theta = \tan^{-1}(0.6) = 31°$$ To the nearest degree

The arrow was shot into the air at 29 feet per second at an angle of elevation of 31°.

Force

Force

Another important vector quantity is *force*. We can loosely define force as a push or a pull.

The most intuitive force in our lives is the force of gravity that pulls us toward the center of the earth. The magnitude of this force is our weight; the direction of this force is always straight down toward the center of the earth.

Although there may be many forces acting on an object at the same time, if the object is stationary, the sum of the forces must be **0**.

Force

This leads us to our next definition.

DEFINITION ■ STATIC EQUILIBRIUM

When an object is stationary (at rest), we say it is in a state of *static equilibrium*. When an object is in this state, the sum of the forces acting on the object must be equal to the zero vector **0**.

Example 5

Danny is 5 years old and weighs 42.0 pounds. He is sitting on a swing when his sister Stacey pulls him and the swing back horizontally through an angle of 30.0° and then stops. Find the tension in the ropes of the swing and the magnitude of the force exerted by Stacey. (Figure 17 is a diagram of the situation.)

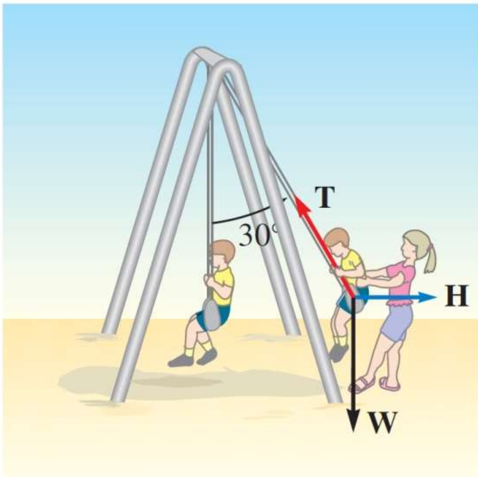

Figure 17

Example 5 – *Solution*

As you can see from Figure 17, there are three forces acting on Danny (and the swing), which we have labeled **W**, **H**, and **T**.

The vector **W** is due to the force of gravity, pulling him toward the center of the earth. Its magnitude is $|\mathbf{W}| = 42.0$ pounds, and its direction is straight down.

The vector **H** represents the force with which Stacey is pulling Danny horizontally, and **T** is the force acting on Danny in the direction of the ropes. We call this force the *tension* in the ropes.

Example 5 – *Solution*

cont'd

If we rearrange the vectors from the diagram in Figure 17, we can get a better picture of the situation.

Since Stacey is holding Danny in the position shown in Figure 17, he is in a state of static equilibrium.

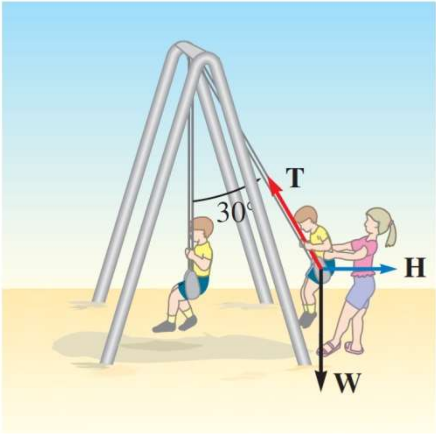

Figure 17

Therefore, the sum of the forces acting on him is **0**. We add the three vectors **W**, **H**, and **T** using the tip-to-tail rule.

The resultant vector, extending from the tail of **W** to the tip of **T**, must have a length of 0.

Example 5 – *Solution*

cont'd

This means that the tip of **T** has to coincide with the tail of **W**, forming a right triangle, as shown in Figure 18.

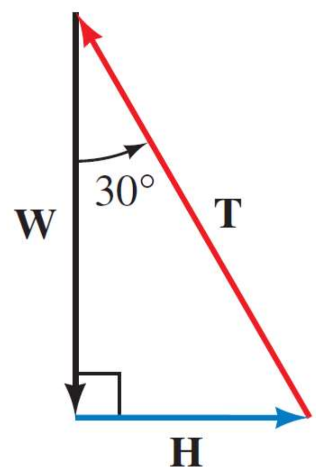

Figure 18

The lengths of the sides of the right triangle shown in Figure 18 are given by the magnitudes of the vectors.

Example 5 – *Solution*

cont'd

We use right triangle trigonometry to find the magnitude of **T**.

$$\cos 30.0^\circ = \frac{|\mathbf{W}|}{|\mathbf{T}|} \qquad \text{Definition of cosine}$$

$$|\mathbf{T}| = \frac{|\mathbf{W}|}{\cos 30.0^\circ} \qquad \text{Solve for } |\mathbf{T}|$$

$$= \frac{42.0}{0.8660} \qquad \text{The magnitude of } \mathbf{W} \text{ is } 42.0$$

$$= 48.5 \text{ lb} \qquad \text{To three significant digits}$$

Next, let's find the magnitude of the force with which Stacey pulls on Danny to keep him in static equilibrium.

Example 5 – *Solution*

cont'd

$$\tan 30.0° = \frac{|\mathbf{H}|}{|\mathbf{W}|}$$ Definition of tangent

$$|\mathbf{H}| = |\mathbf{W}| \tan 30.0°$$ Solve for $|\mathbf{H}|$

$$= 42.0(0.5774)$$

$$= 24.2 \text{ lb}$$ To three significant digits

Stacey must pull horizontally with a force of magnitude 24.2 pounds to hold Danny at an angle of 30.0° from vertical.

Work

Work

One application of vectors that is related to the concept of force is *work*. Intuitively, work is a measure of the "effort" expended when moving an object by applying a force to it.

For example, if you have ever had to push a stalled automobile or lift a heavy object, then you have experienced work.

Assuming the object moves along a straight line, then the work is calculated by finding the component of the force parallel to this line and multiplying it by the distance the object is moved. A common unit of measure for work is the foot-pound (ft-lb).

Work

DEFINITION ■ WORK

If a constant force **F** is applied to an object and moves the object in a straight line a distance d, then the *work* W performed by the force is

$$W = (|\mathbf{F}| \cos \theta) \cdot d$$

where θ is the angle between the force **F** and the line of motion of the object.

Our next example illustrates this situation.

Example 6

A shipping clerk pushes a heavy package across the floor. He applies a force of 64 pounds in a downward direction, making an angle of 35° with the horizontal. If the package is moved 25 feet, how much work is done by the clerk?

Solution:

A diagram of the problem is shown in Figure 19.

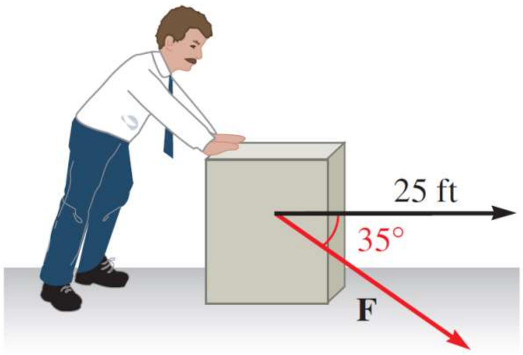

Figure 19

Example 6 – *Solution*

cont'd

Because the package moves in a horizontal direction, and not in the direction of the force, we must first find the amount of force that is directed horizontally.

This will be the magnitude of the horizontal vector component of **F**.

$$|\mathbf{F}_x| = |\mathbf{F}| \cos 35° = 64 \cos 35° \text{ lb}$$

Example 6 – *Solution*

cont'd

To find the work, we multiply this value by the distance the package moves.

$$\text{Work} = (64 \cos 35°)(25)$$

$$= 1{,}300 \text{ ft-lb} \qquad \text{To two significant digits}$$

1,300 foot-pounds of work are performed by the shipping clerk in moving the package.