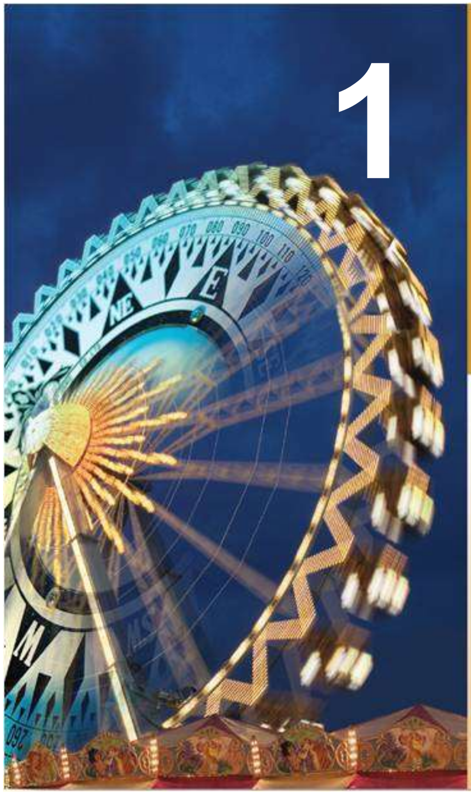

1 The Six Trigonometric Functions

SECTION 1.5

More on Identities

Learning Objectives

1. Write an expression in terms of sines and cosines.
2. Simplify an expression containing trigonometric functions.
3. Use a trigonometric substitution to simplify a radical expression.
4. Verify an equation is an identity.

Example 1

Write tan θ in terms of sin θ.

Solution:

When we say we want tan θ written in terms of sin θ, we mean that we want to write an expression that is equivalent to tan θ but involves no trigonometric function other than sin θ.

Let's begin by using a ratio identity to write tan θ in terms of sin θ and cos θ.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Example 1 – *Solution*

cont'd

Now we need to replace cos θ with an expression involving only sin θ. Since $\cos \theta = \pm\sqrt{1 - \sin^2 \theta}$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$= \frac{\sin \theta}{\pm\sqrt{1 - \sin^2 \theta}}$$

$$= \pm\frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$$

This last expression is equivalent to tan θ and is written in terms of sin θ only. (In a problem like this, it is okay to include numbers and algebraic symbols with sin θ.)

Example 4

Multiply $(\sin\theta + 2)(\sin\theta - 5)$.

Solution:

We multiply these two expressions in the same way we would multiply $(x + 2)(x - 5)$. (In some algebra books, this kind of multiplication is accomplished using the FOIL method.)

$$\begin{aligned}(\sin\theta + 2)(\sin\theta - 5) &= \sin\theta\sin\theta - 5\sin\theta + 2\sin\theta - 10 \\ &= \sin^2\theta - 3\sin\theta - 10\end{aligned}$$

Example 5

Simplify the expression $\sqrt{x^2 + 9}$ as much as possible after substituting 3 tan θ for *x*.

Solution:

Our goal is to write the expression $\sqrt{x^2 + 9}$ without a square root by first making the substitution $x = 3 \tan \theta$.

If $x = 3 \tan \theta$

then the expression $\sqrt{x^2 + 9}$

becomes

$$\sqrt{(3 \tan \theta)^2 + 9} = \sqrt{9 \tan^2 \theta + 9}$$

$$= \sqrt{9(\tan^2 \theta + 1)}$$

$$= \sqrt{9 \sec^2 \theta}$$

$$= 3 \, |\sec \theta|$$

More on Identities

Note 1: We must use the absolute value symbol unless we know that sec θ is positive. Remember, in algebra, $\sqrt{a^2} = a$ only when *a* is positive or zero. If it is possible that *a* is negative, then $\sqrt{a^2} = |a|$.

Note 2: After reading through Example 5, you may be wondering if it is mathematically correct to make the substitution $x = 3 \tan \theta$. After all, *x* can be any real number because $x^2 + 9$ will always be positive.

Example 7

Prove the identity $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$.

Solution:

Let's agree to prove the identities in this section, and the problem set that follows, by transforming the left side into the right side. In this case, we begin by expanding $(\sin\theta + \cos\theta)^2$. (Remember from algebra, $(a + b)^2 = a^2 + 2ab + b^2$.)

$$
\begin{aligned}
(\sin\theta + \cos\theta)^2 &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \\
&= (\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta && \text{Rearrange terms} \\
&= 1 + 2\sin\theta\cos\theta && \text{Pythagorean identity}
\end{aligned}
$$

More on Identities

We should mention that the ability to prove identities in trigonometry is not always obtained immediately.

It usually requires a lot of practice. The more you work at it, the better you will become at it.