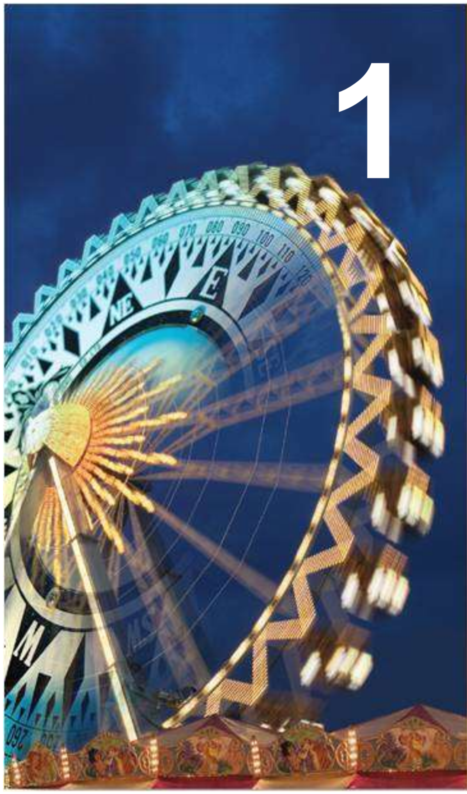

1 The Six Trigonometric Functions

SECTION 1.4

Introduction to Identities

Objectives

1. Find the value of a trigonometric function using a reciprocal identity.
2. Find the value of a trigonometric function using a ratio identity.
3. Evaluate a trigonometric function raised to an exponent.
4. Use a Pythagorean identity to find the value of a trigonometric function.

Reciprocal Identities

Reciprocal Identities

Our definition for the sine and cosecant functions indicate that they are reciprocals; that is,

$$\csc\theta = \frac{1}{\sin\theta} \qquad \text{because} \qquad \frac{1}{\sin\theta} = \frac{1}{y/r} = \frac{r}{y} = \csc\theta$$

Note: We can also write this same relationship between sin θ and csc θ in another form as

$$\sin\theta = \frac{1}{\csc\theta} \qquad \text{because} \qquad \frac{1}{\csc\theta} = \frac{1}{r/y} = \frac{y}{r} = \sin\theta$$

The first identity we wrote, csc θ = 1/sin θ, is the basic identity. The second one, sin θ = 1/csc θ, is an equivalent form of the first.

Reciprocal Identities

Table 1 lists three basic reciprocal identities and their common equivalent forms.

Reciprocal Identities	**Equivalent Forms**
$\csc\theta = \dfrac{1}{\sin\theta}$	$\sin\theta = \dfrac{1}{\csc\theta}$
$\sec\theta = \dfrac{1}{\cos\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$
$\cot\theta = \dfrac{1}{\tan\theta}$	$\tan\theta = \dfrac{1}{\cot\theta}$

Table 1

Reciprocal Identities

Example

If $\sin\theta = \dfrac{3}{5}$, then $\csc\theta = \dfrac{5}{3}$, because

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{3/5} = \frac{5}{3}$$

Ratio Identities

Ratio Identities

There are two ratio identities, one for tan θ and one for cot θ (see Table 2).

Ratio Identities		
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	because	$\dfrac{\sin\theta}{\cos\theta} = \dfrac{y/r}{x/r} = \dfrac{y}{x} = \tan\theta$
$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	because	$\dfrac{\cos\theta}{\sin\theta} = \dfrac{x/r}{y/r} = \dfrac{x}{y} = \cot\theta$

Table 2

Example 7

If sin θ = –3/5 and cos θ = 4/5, find tan θ and cot θ.

Solution:

Using the ratio identities, we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-3/5}{4/5} = -\frac{3}{5}\cdot\frac{5}{4} = -\frac{3}{4}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{4/5}{-3/5} = \frac{4}{5}\cdot\left(-\frac{5}{3}\right) = -\frac{4}{3}$$

Example 7 – *Solution*

cont'd

Note: Once we found tan θ, we could have used a reciprocal identity to find cot θ.

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-3/4} = -\frac{4}{3}$$

Ratio Identities

NOTATION

The notation $\sin^2 \theta$ is a shorthand notation for $(\sin \theta)^2$. It indicates we are to square the number that is the sine of θ.

Example

$$\text{If } \sin \theta = \frac{3}{5}, \text{ then } \sin^2 \theta = \left(\frac{3}{5}\right)^2 = \frac{9}{25}.$$

Pythagorean Identities

Pythagorean Identities

We start with the relationship among x, y, and r as given in the definition of sin θ and cos θ.

We summarize the identities in Table 3.

Pythagorean Identities	Equivalent Forms
$\cos^2\theta + \sin^2\theta = 1$	$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$ $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$
$1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$	

Table 3

Example 11

If cos θ = 1/2 and θ terminates in QIV, find the remaining trigonometric ratios for θ.

Solution:

The first, and easiest, ratio to find is sec θ because it is the reciprocal of cos θ.

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{1/2} = 2$$

Next we find sin θ. Using one of the equivalent forms of our Pythagorean identity, we have

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

Example 11 – *Solution*

cont'd

Because θ terminates in QIV, sin θ will be negative. This gives us

$$\sin\theta = -\sqrt{1-\cos^2\theta} \qquad \text{Negative sign because } \theta \in \text{QIV}$$

$$= -\sqrt{1-\left(\frac{1}{2}\right)^2} \qquad \text{Substitute } \frac{1}{2} \text{ for } \cos\theta$$

$$= -\sqrt{1-\frac{1}{4}} \qquad \text{Square } \frac{1}{2} \text{ to get } \frac{1}{4}$$

Example 11 – *Solution*

cont'd

$$= -\sqrt{\frac{3}{4}}$$

Subtract

$$= -\frac{\sqrt{3}}{2}$$

Take the square root of the numerator and denominator separately

Now that we have sin θ and cos θ, we can find tan θ by using a ratio identity.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{3}/2}{1/2} = -\frac{\sqrt{3}}{2}\cdot\frac{2}{1} = -\sqrt{3}$$

Example 11 – *Solution*

cont'd

Cot θ and csc θ are the reciprocals of tan θ and sin θ, respectively. Therefore,

$$\cot\theta = \frac{1}{\tan\theta} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

Here are all six ratios together:

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \csc\theta = -\frac{2\sqrt{3}}{3}$$

Example 11 – *Solution*

cont'd

$$\cos\theta = \frac{1}{2} \qquad \sec\theta = 2$$

$$\tan\theta = -\sqrt{3} \qquad \cot\theta = -\frac{\sqrt{3}}{3}$$