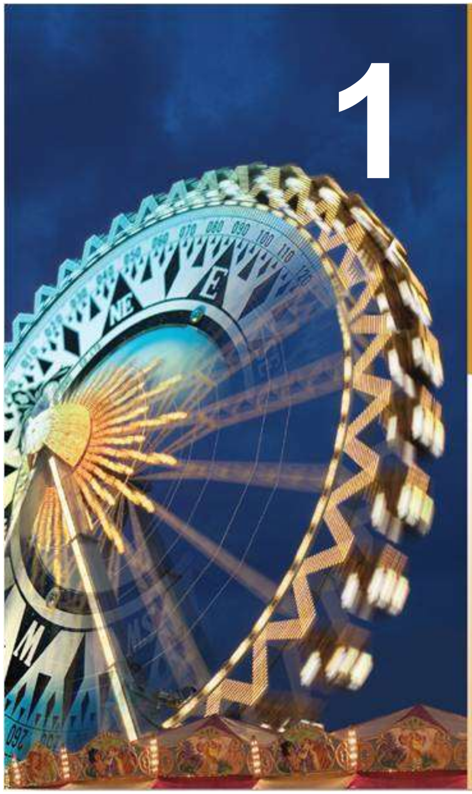


1 The Six Trigonometric Functions

SECTION 1.2

The Rectangular Coordinate System

Objectives

1. Verify a point lies on the graph of the unit circle.
2. Find the distance between two points.
3. Draw an angle in standard position.
4. Find an angle that is coterminal with a given angle.

The Rectangular Coordinate System

The rectangular coordinate system allows us to connect algebra and geometry by associating geometric shapes with algebraic equations.

For example, every nonvertical straight line (a geometric concept) can be paired with an equation of the form $y = mx + b$ (an algebraic concept), where m and b are real numbers, and x and y are variables that we associate with the axes of a coordinate system.

The Rectangular Coordinate System

The rectangular (or Cartesian) coordinate system is shown in Figure 1.

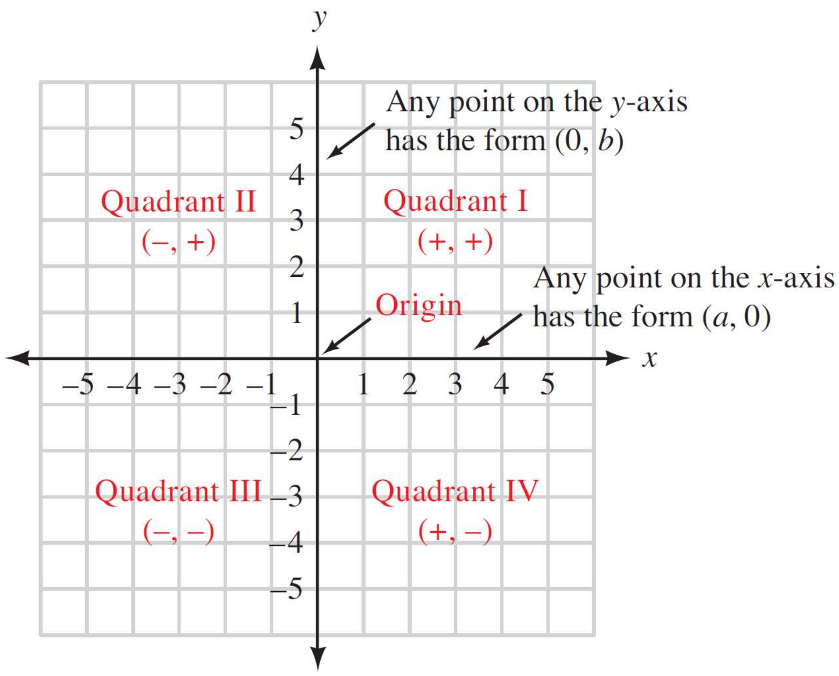


Figure 1

The axes divide the plane into four *quadrants* that are numbered I through IV in a counterclockwise direction.

The Rectangular Coordinate System

Looking at Figure 1, we see that any point in quadrant I will have both coordinates positive; that is, (+, +). In quadrant II, the form is (–, +). In quadrant III, the form is (–, –), and in quadrant IV it is (+, –).

Also, any point on the x-axis will have a y-coordinate of 0 (it has no vertical displacement), and any point on the y-axis will have an x-coordinate of 0 (no horizontal displacement).

Graphing Lines

Example 1

Graph the line $y = \frac{3}{2}x$.

Solution:

Because the equation of the line is written in slope-intercept form, we see that the slope of the line is $\frac{3}{2}$ = 1.5 and the *y*-intercept is 0.

To graph the line, we begin at the origin and use the slope to locate a second point.

For every unit we traverse to the right, the line will rise 1.5 units. If we traverse 2 units to the right, the line will rise 3 units, giving us the point (2, 3).

Example 1 – *Solution*

cont'd

Or, if we traverse 3 units to the right, the line will rise 4.5 units yielding the point (3, 4.5). The graph of the line is shown in Figure 2.

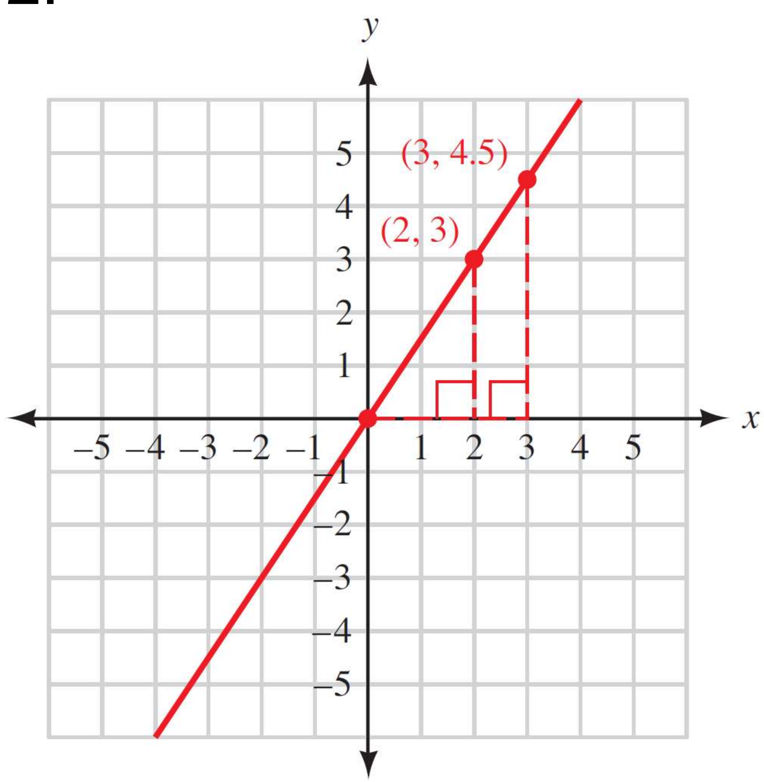


Figure 2

Graphing Parabolas

Graphing Parabolas

A parabola that opens up or down can be described by an equation of the form

$$y = a(x - h)^2 + k$$

Likewise, any equation of this form will have a graph that is a parabola. The highest or lowest point on the parabola is called the vertex.

The coordinates of the vertex are (h, k). The value of a determines how wide or narrow the parabola will be and whether it opens upward or downward.

Example 2

At the 1997 Washington County Fair in Oregon, David Smith, Jr., The Bullet, was shot from a cannon. As a human cannonball, he reached a height of 70 feet before landing in a net 160 feet from the cannon. Sketch the graph of his path, and then find the equation of the graph.

Solution:

We assume that the path taken by the human cannonball is a parabola.

If the origin of the coordinate system is at the opening of the cannon, then the net that catches him will be at 160 on the x-axis.

Example 2 – *Solution*

cont'd

Figure 5 shows a graph of this path.

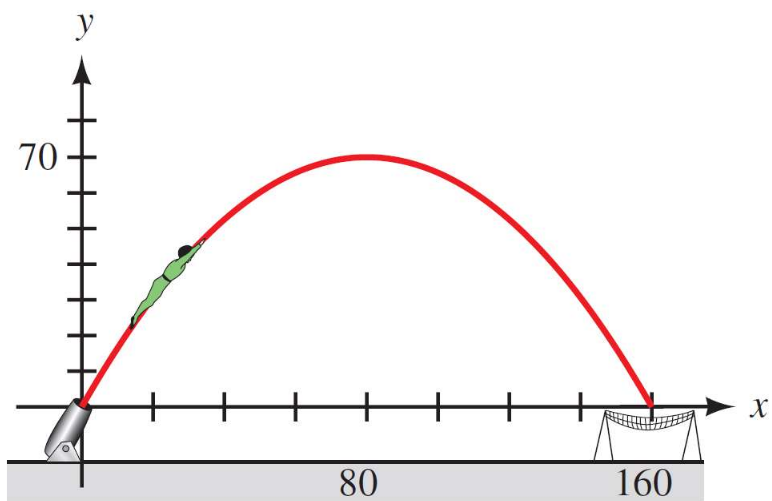


Figure 5

Because the curve is a parabola, we know that the equation will have the form

$$y = a(x - h)^2 + k$$

Example 2 – *Solution*

cont'd

Because the vertex of the parabola is at (80, 70), we can fill in two of the three constants in our equation, giving us

$$y = a(x - 80)^2 + 70$$

To find *a* we note that the landing point will be (160, 0). Substituting the coordinates of this point into the equation, we solve for *a*.

$$0 = a(160 - 80)^2 + 70$$

$$0 = a(80)^2 + 70$$

$$0 = 6400a + 70$$

$$a = -\frac{70}{6400} = -\frac{7}{640}$$

Example 2 – *Solution*

cont'd

The equation that describes the path of the human cannonball is

$$y = -\frac{7}{640}(x - 80)^2 + 70 \text{ for } 0 \leq x \leq 160$$

The Distance Formula

The Distance Formula

Our next definition gives us a formula for finding the distance between any two points on the coordinate system.

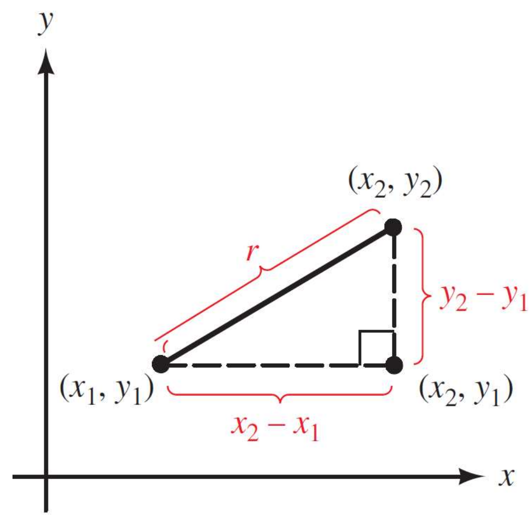
THE DISTANCE FORMULA

The distance between any two points (x_1, y_1) and (x_2, y_2) in a rectangular coordinate system is given by the formula

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The Distance Formula

The distance formula can be derived by applying the Pythagorean Theorem to the right triangle in Figure 8. Because r is a distance, $r \geq 0$.



$$r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Figure 8

Example 3

Find the distance between the points (–1, 5) and (2, 1).

Solution:

It makes no difference which of the points we call (x_1, y_1) and which we call (x_2, y_2) because this distance will be the same between the two points regardless (Figure 9).

$$r = \sqrt{(2 - (-1))^2 + (1 - 5)^2} = \sqrt{3^2 + (-4)^2}$$

$$= \sqrt{9 + 16}$$

$$= \sqrt{25}$$

$$= 5$$

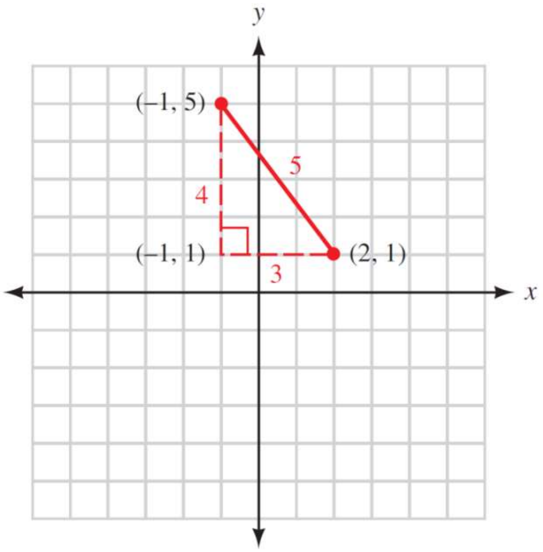


Figure 9

Circles

Circles

A *circle* is defined as the set of all points in the plane that are a fixed distance from a given fixed point. The fixed distance is the *radius* of the circle, and the fixed point is called the *center*.

If we let $r > 0$ be the radius, (h, k) the center, and (x, y) represent any point on the circle.

Circles

Then (x, y) is r units from (h, k) as Figure 11 illustrates.

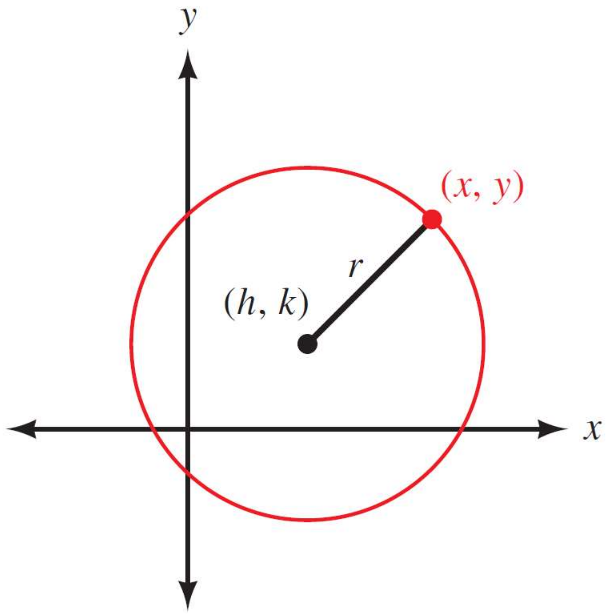


Figure 11

Applying the distance formula, we have

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

Circles

Squaring both sides of this equation gives the formula for a circle.

EQUATION OF A CIRCLE

The equation of a circle with center (h, k) and radius $r > 0$ is given by the formula

$$(x - h)^2 + (y - k)^2 = r^2$$

If the center is at the origin so that $(h, k) = (0, 0)$, this simplifies to

$$x^2 + y^2 = r^2$$

Example 5

Verify that the points $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ both lie on a circle of radius 1 centered at the origin.

Solution:

Because $r = 1$, the equation of the circle is $x^2 + y^2 = 1$. We check each point by showing that the coordinates satisfy the equation.

If $\quad x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$

then $\quad x^2 + y^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2$

If $\quad x = -\frac{\sqrt{3}}{2}$ and $y = -\frac{1}{2}$

then $\quad x^2 + y^2 = \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2$

Example 5 – *Solution*

cont'd

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

$$= \frac{3}{4} + \frac{1}{4}$$

$$= 1$$

The graph of the circle and the two points are shown in Figure 12.

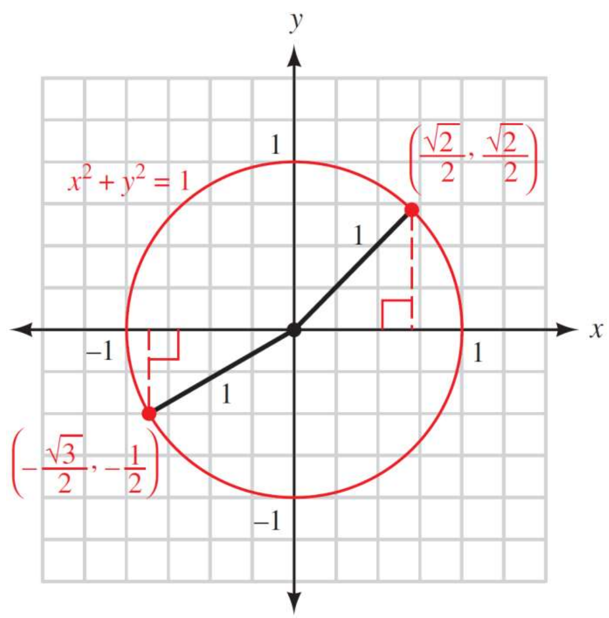


Figure 12

Circles

The circle $x^2 + y^2 = 1$ from Example 5 is called the *unit circle* because its radius is 1.

Angles in Standard Position

Angles in Standard Position

DEFINITION

An angle is said to be in *standard position* if its initial side is along the positive x-axis and its vertex is at the origin.

Example 6

Draw an angle of 45° in standard position and find a point on the terminal side.

Solution:

If we draw 45° in standard position, we see that the terminal side is along the line $y = x$ in quadrant I (Figure 16).

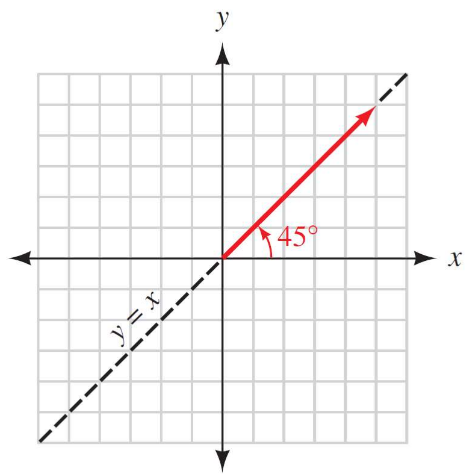


Figure 16

Example 6 – *Solution*

cont'd

Because the terminal side of 45° lies along the line $y = x$ in the first quadrant, any point on the terminal side will have positive coordinates that satisfy the equation $y = x$.

Here are some of the points that do just that.

$$(1, 1) \qquad (2, 2) \qquad (3, 3) \qquad (\sqrt{2}, \sqrt{2}) \qquad \left(\frac{1}{2}, \frac{1}{2}\right) \qquad \left(\frac{7}{8}, \frac{7}{8}\right)$$

Angles in Standard Position

VOCABULARY

If angle θ is in standard position and the terminal side of θ lies in quadrant I, then we say θ lies in quadrant I and we abbreviate it like this:

$$\theta \in \text{QI}$$

Likewise, $\theta \in \text{QII}$ means θ is in standard position with its terminal side in quadrant II.

If the terminal side of an angle in standard position lies along one of the axes, then that angle is called a *quadrantal angle*.

Angles in Standard Position

For example, an angle of 90° drawn in standard position would be a quadrantal angle, because the terminal side would lie along the positive *y*-axis. Likewise, 270° in standard position is a quadrantal angle because the terminal side would lie along the negative *y*-axis (Figure 17).

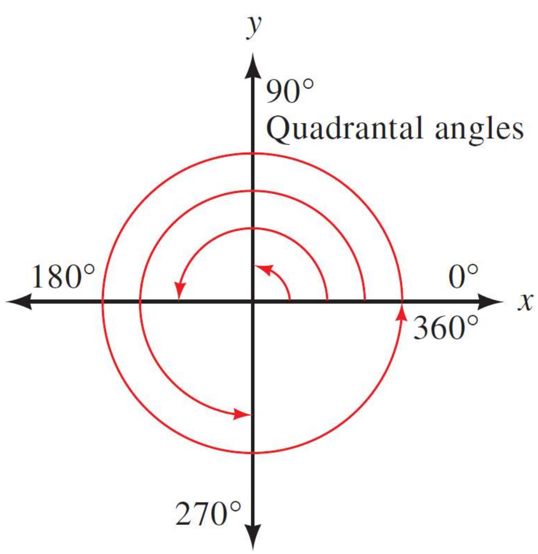


Figure 17

Angles in Standard Position

Two angles in standard position with the same terminal side are called *coterminal angles*. Figure 18 shows that 60° and –300° are coterminal angles when they are in standard position.

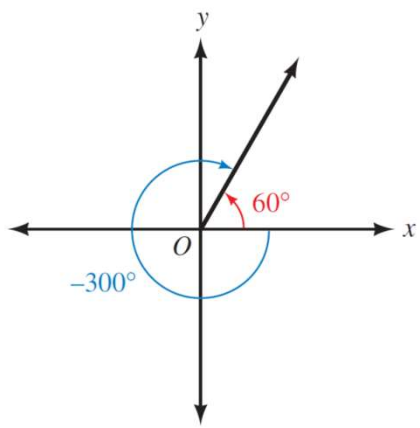


Figure 18

Notice that these two angles differ by 360°. That is, 60° – (–300°) = 360°. Coterminal angles always differ from each other by some multiple of 360°.

Example 7

Draw –90° in standard position and find two positive angles and two negative angles that are coterminal with –90°.

Solution:

Figure 19 shows –90° in standard position.

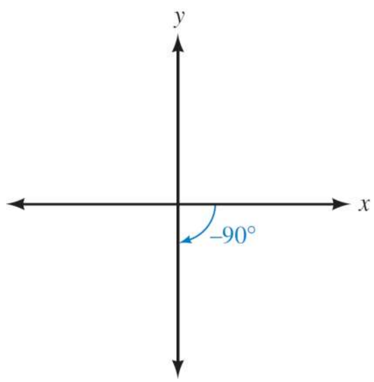


Figure 19

To find a coterminal angle, we must traverse a full revolution in the positive direction or the negative direction.

Example 7 – *Solution*

cont'd

One revolution in the positive direction:	$-90° + 360° = 270°$
A second revolution in the positive direction:	$270° + 360° = 630°$
One revolution in the negative direction:	$-90° - 360° = -450°$
A second revolution in the negative direction:	$-450° - 360° = -810°$

Thus, 270° and 630° are two positive angles coterminal with –90° and –450° and –810° are two negative angles coterminal with –90°.

Example 7 – *Solution*

cont'd

Figures 20 and 21 show two of these angles.

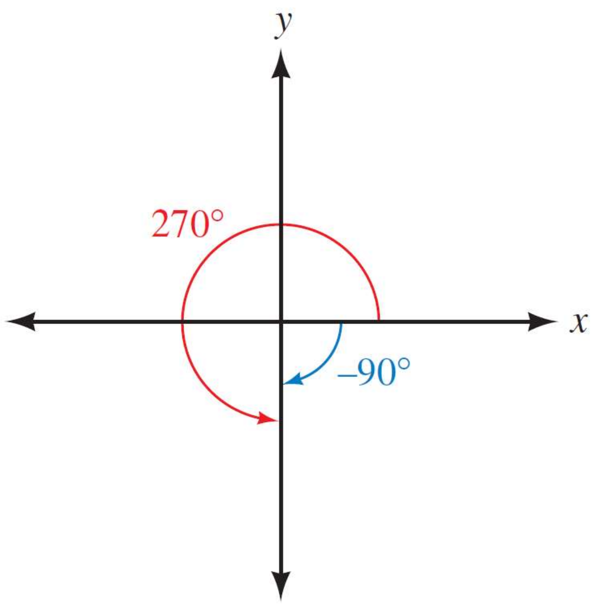


Figure 20

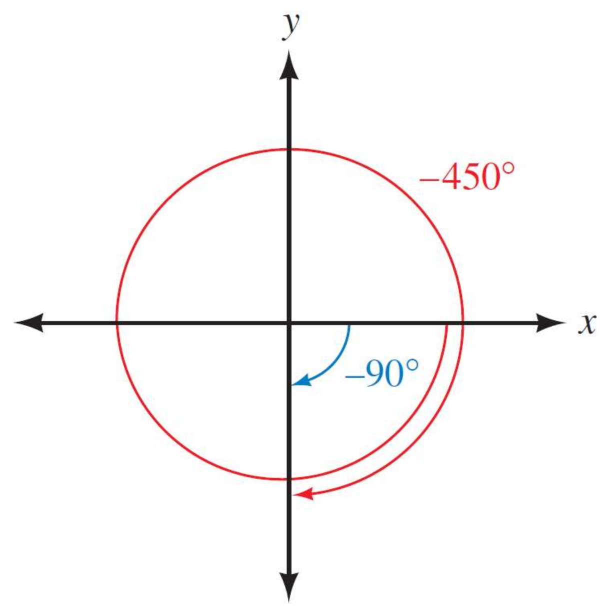


Figure 21