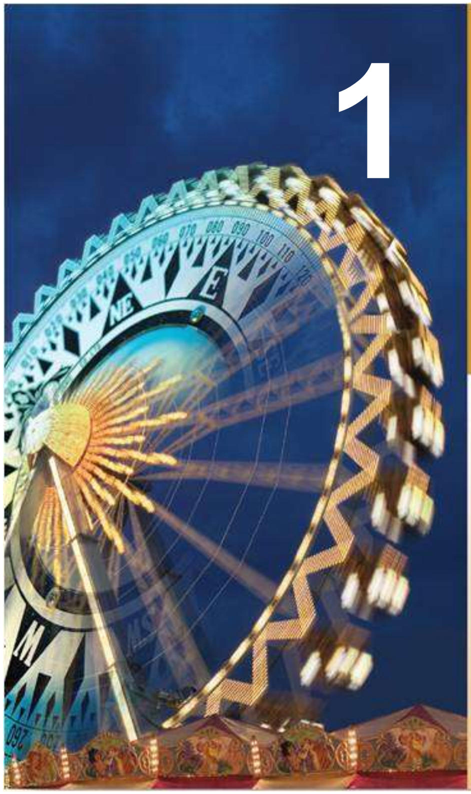


1 The Six Trigonometric Functions

SECTION 1.1

Angles, Degrees, and Special Triangles

Learning Objectives

1. Compute the complement and supplement of an angle.

2. Use the Pythagorean Theorem to find the third side of a right triangle.

3. Find the other two sides of a 30°–60°–90° or 45°–45°–90° triangle given one side.

4. Solve a real-life problem using the special triangle relationships.

Angles in General

Angles in General

An angle is formed by two rays with the same end point. The common end point is called the *vertex* of the angle, and the rays are called the *sides* of the angle.

In Figure 1 the vertex of angle θ (theta) is labeled *O*, and *A* and *B* are points on each side of θ.

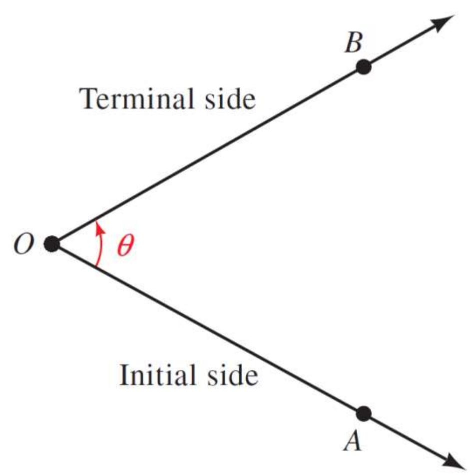


Figure 1

Angles in General

Angle θ can also be denoted by *AOB*, where the letter associated with the vertex is written between the letters associated with the points on each side.

We can think of θ as having been formed by rotating side *OA* about the vertex to side *OB*.

In this case, we call side *OA* the *initial side* of θ and side *OB* the *terminal side* of θ.

Angles in General

When the rotation from the initial side to the terminal side takes place in a counterclockwise direction, the angle formed is considered a *positive angle*.

If the rotation is in a clockwise direction, the angle formed is a *negative angle* (Figure 2).

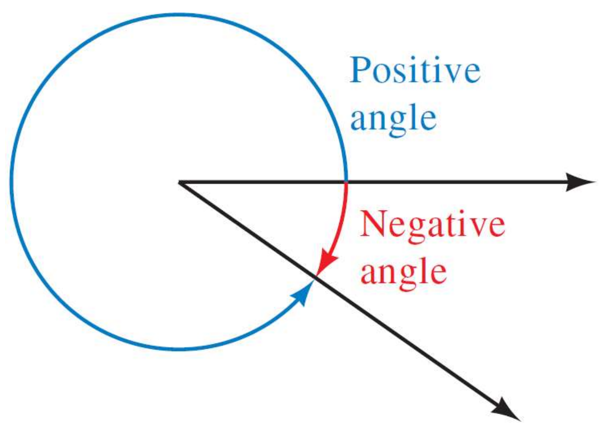


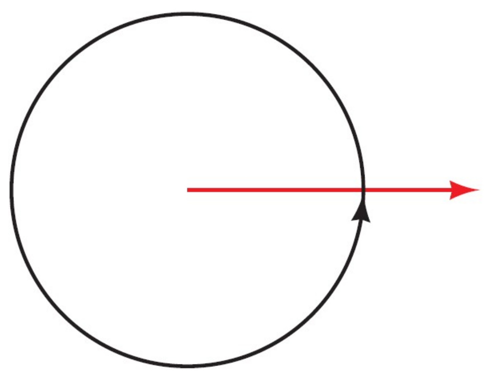
Figure 2

Degree Measure

Degree Measure

One way to measure the size of an angle is with degree measure.

The angle formed by rotating a ray through one complete revolution has a measure of 360 degrees, written 360° (Figure 3).



One complete revolution = 360°

Figure 3

Degree Measure

One degree (1°), then, is 1/360 of a full rotation. Likewise, 180° is one-half of a full rotation, and 90° is half of that (or a quarter of a rotation).

Angles that measure 90° are called *right angles*, while angles that measure 180° are called *straight angles*.

Degree Measure

Angles that measure between 0° and 90° are called *acute angles*, while angles that measure between 90° and 180° are called *obtuse angles* (see Figure 4).

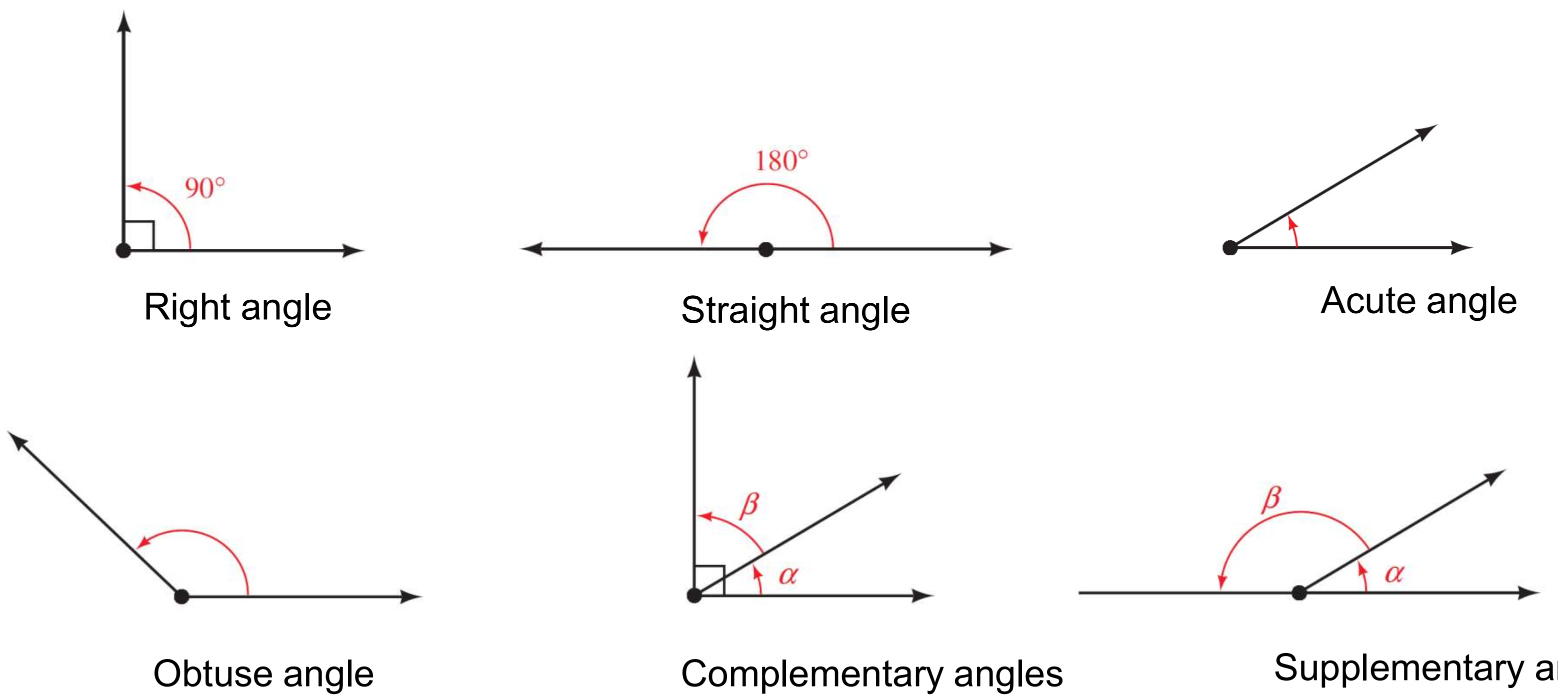


Figure 4

Degree Measure

If two angles have a sum of 90°, then they are called *complementary angles*, and we say each is the *complement* of the other.

Two angles with a sum of 180° are called *supplementary angles*.

Example 1

Give the complement and the supplement of each angle.

a. 40° **b.** 110° **c.** θ

Solution:

a. The complement of 40° is 50° since $40^\circ + 50^\circ = 90^\circ$.

The supplement of 40° is 140° since $40^\circ + 140^\circ = 180^\circ$.

b. The complement of 110° is -20° since $110^\circ + (-20^\circ) = 90^\circ$.

The supplement of 110° is 70° since $110^\circ + 70^\circ = 180^\circ$.

Example 1 – *Solution*

cont'd

c. The complement of θ is $90° - \theta$ since $\theta + (90° - \theta) = 90°$.

The supplement of θ is $180° - \theta$ since $\theta + (180° - \theta) = 180°$.

Triangles

Triangles

A triangle is a three-sided polygon. Every triangle has three sides and three angles. We denote the angles (or vertices) with uppercase letters and the lengths of the sides with lowercase letters, as shown in Figure 5.

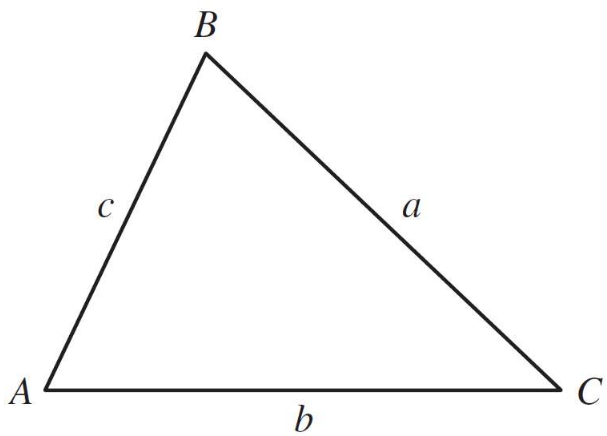


Figure 5

It is standard practice in mathematics to label the sides and angles so that *a* is opposite *A*, *b* is opposite *B*, and *c* is opposite *C*.

Triangles

There are different types of triangles that are named according to the relative lengths of their sides or angles (Figure 6).

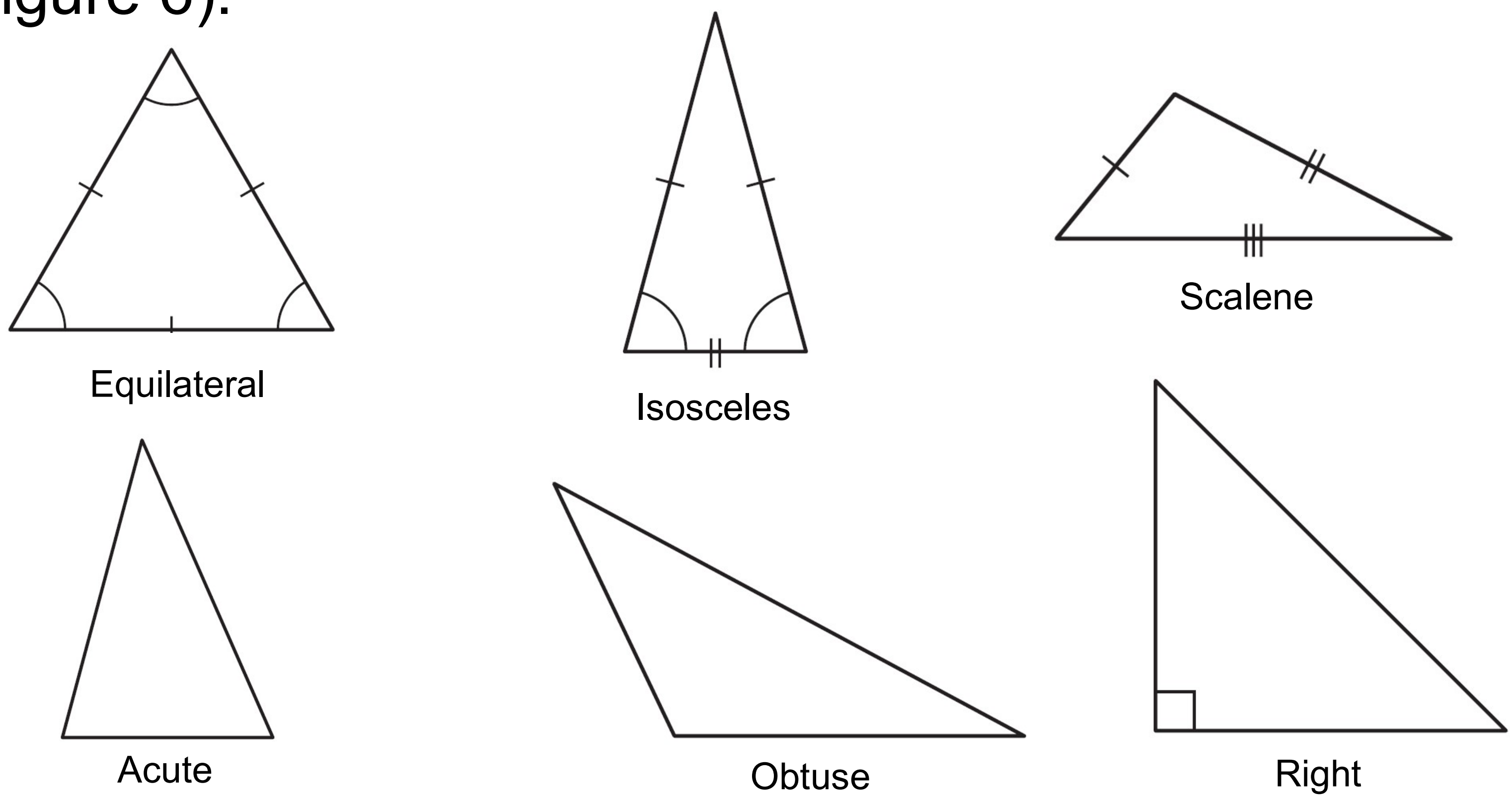


Figure 6

Triangles

In an *equilateral triangle*, all three sides are of equal length and all three angles are equal.

An *isosceles triangle* has two equal sides and two equal angles. If all the sides and angles are different, the triangle is called *scalene*.

In an *acute triangle*, all three angles are acute. An *obtuse triangle* has exactly one obtuse angle, and a *right triangle* has one right angle.

Special Triangles

Special Triangles

Right triangles are very important to the study of trigonometry. In every right triangle, the longest side is called the *hypotenuse*, and it is always opposite the right angle.

The other two sides are called the *legs* of the right triangle. Because the sum of the angles in any triangle is 180°, the other two angles in a right triangle must be complementary, acute angles.

The Pythagorean Theorem gives us the relationship that exists among the sides of a right triangle.

Special Triangles

First we state the theorem.

PYTHAGOREAN ■ THEOREM

In any right triangle, the square of the length of the longest side (called the hypotenuse) is equal to the sum of the squares of the lengths of the other two sides (called legs).

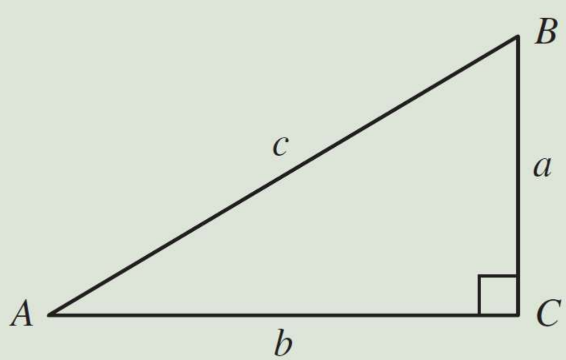


Figure 7

If $C = 90°$,
then $c^2 = a^2 + b^2$.

A Proof of the Pythagorean Theorem

A Proof of the Pythagorean Theorem

There are many ways to prove the Pythagorean Theorem. The method that we are offering here is based on the diagram shown in Figure 8 and the formula for the area of a triangle.

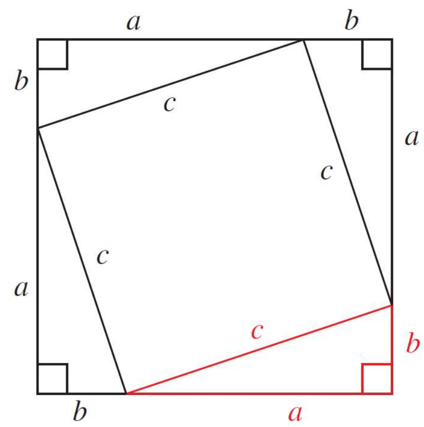


Figure 8

Figure 8 is constructed by taking the right triangle in the lower right corner and repeating it three times so that the final diagram is a square in which each side has length $a + b$.

A Proof of the Pythagorean Theorem

To derive the relationship between *a*, *b*, and *c*, we simply notice that the area of the large square is equal to the sum of the areas of the four triangles and the inner square.

In symbols we have

$$\underset{\text{Area of large square}}{(a+b)^2} = \underset{\text{Area of four triangles}}{4\left(\frac{1}{2}ab\right)} + \underset{\text{Area of inner square}}{c^2}$$

A Proof of the Pythagorean Theorem

We expand the left side using the formula for the square of a binomial, from algebra.

We simplify the right side by multiplying 4 by $\frac{1}{2}$.

$$a^2 + 2ab + b^2 = 2ab + c^2$$

Adding –2*ab* to each side, we have the relationship we are after:

$$a^2 + b^2 = c^2$$

Example 2

Solve for *x* in the right triangle in Figure 9.

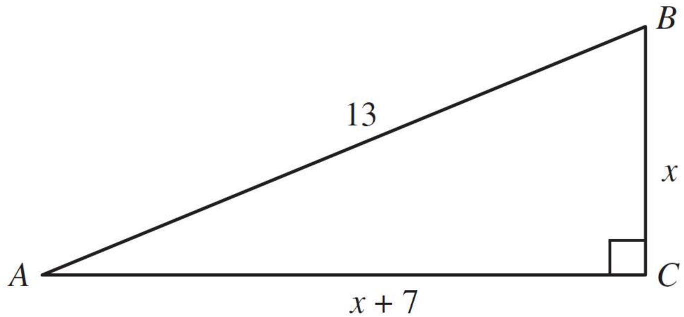


Figure 9

Solution:

Applying the Pythagorean Theorem gives us a quadratic equation to solve.

$$(x + 7)^2 + x^2 = 13^2$$

$$x^2 + 14x + 49 + x^2 = 169 \qquad \text{Expand } (x + 7)^2 \text{ and } 13^2$$

Example 2 – *Solution*

cont'd

$$2x^2 + 14x + 49 = 169 \qquad \text{Combine similar terms}$$

$$2x^2 + 14x - 120 = 0 \qquad \text{Add } -169 \text{ to both sides}$$

$$x^2 + 7x - 60 = 0 \qquad \text{Divide both sides by 2}$$

$$(x - 5)(x + 12) = 0 \qquad \text{Factor the left side}$$

$$x - 5 = 0 \quad \text{or} \quad x + 12 = 0 \qquad \text{Set each factor to 0}$$

$$x = 5 \quad \text{or} \quad x = -12$$

Our only solution is $x = 5$. We cannot use $x = -12$ because x is the length of a side of triangle *ABC* and therefore cannot be negative.

A Proof of the Pythagorean Theorem

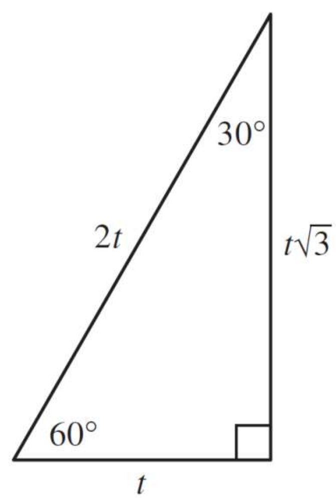
Note: The lengths of the sides of the triangle in Example 2 are 5, 12, and 13.

Whenever the three sides in a right triangle are natural numbers, those three numbers are called a *Pythagorean triple*.

A Proof of the Pythagorean Theorem

THE 30°–60°–90° TRIANGLE

In any right triangle in which the two acute angles are 30° and 60°, the longest side (the hypotenuse) is always twice the shortest side (the side opposite the 30° angle), and the side of medium length (the side opposite the 60° angle) is always $\sqrt{3}$ times the shortest side (Figure 13).



30°–60°–90°

Figure 13

A Proof of the Pythagorean Theorem

Note: The shortest side t is opposite the smallest angle 30°.

The longest side $2t$ is opposite the largest angle 90°.

Example 5

A ladder is leaning against a wall. The top of the ladder is 4 feet above the ground and the bottom of the ladder makes an angle of 60° with the ground (Figure 16). How long is the ladder, and how far from the wall is the bottom of the ladder?

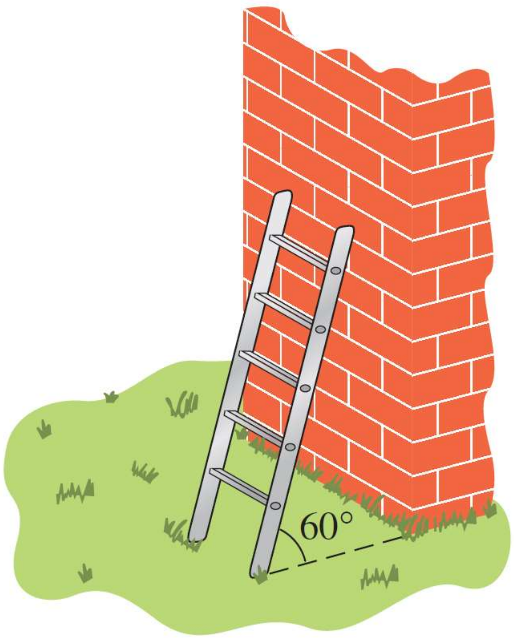


Figure 16

Example 5 – *Solution*

The triangle formed by the ladder, the wall, and the ground is a 30°–60°–90° triangle. If we let *x* represent the distance from the bottom of the ladder to the wall, then the length of the ladder can be represented by 2*x*.

The distance from the top of the ladder to the ground is $x\sqrt{3}$, since it is opposite the 60° angle (Figure 17).

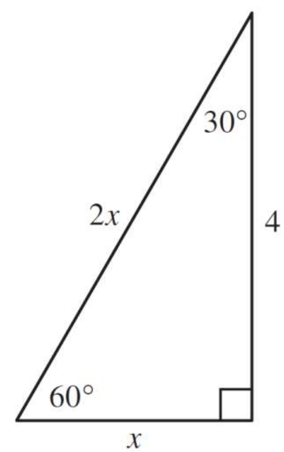


Figure 17

Example 5 – *Solution*

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It is also given as 4 feet.

Therefore,

$$x\sqrt{3} = 4$$

$$x = \frac{4}{\sqrt{3}}$$

$$= \frac{4\sqrt{3}}{3}$$

Rationalize the denominator by multiplying the numerator and denominator by $\sqrt{3}$.

Example 5 – *Solution*

cont'd

The distance from the bottom of the ladder to the wall, *x*, is $4\sqrt{3}/3$ feet, so the length of the ladder, 2*x*, must be $8\sqrt{3}/3$ feet. Note that these lengths are given in exact values.

If we want a decimal approximation for them, we can replace $\sqrt{3}$ with 1.732 to obtain

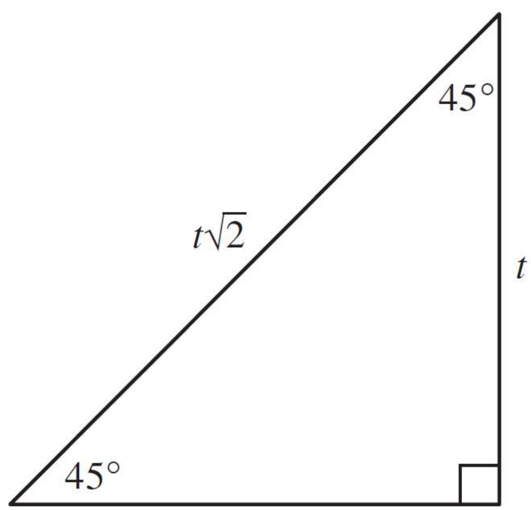
$$\frac{4\sqrt{3}}{3} \approx \frac{4(1.732)}{3} = 2.309 \text{ ft}$$

$$\frac{8\sqrt{3}}{3} \approx \frac{8(1.732)}{3} = 4.619 \text{ ft}$$

A Proof of the Pythagorean Theorem

THE 45°–45°–90° TRIANGLE

If the two acute angles in a right triangle are both 45°, then the two shorter sides (the legs) are equal and the longest side (the hypotenuse) is $\sqrt{2}$ times as long as the shorter sides. That is, if the shorter sides are of length t, then the longest side has length $t\sqrt{2}$ (Figure 18).



45°–45°–90°

Figure 18

Example 6

A 10-foot rope connects the top of a tent pole to the ground. If the rope makes an angle of 45° with the ground, find the length of the tent pole (Figure 19).

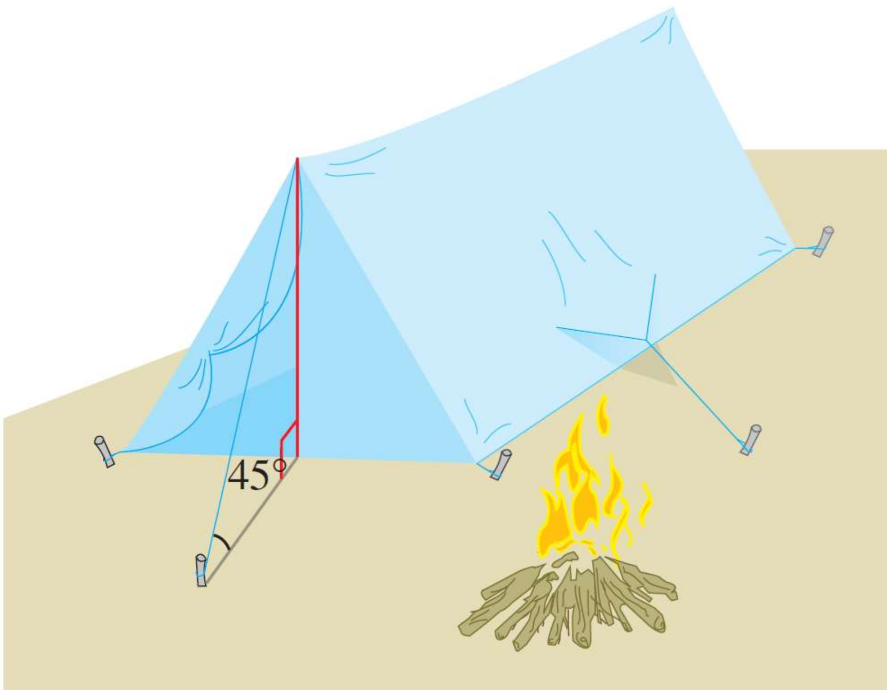


Figure 19

Example 6 – *Solution*

Assuming that the tent pole forms an angle of 90° with the ground, the triangle formed by the rope, tent pole, and the ground is a 45°–45°–90° triangle (Figure 20).

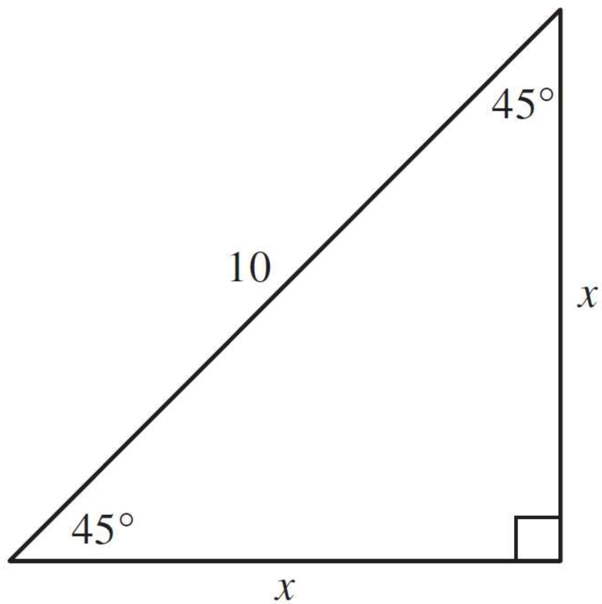


Figure 20

Example 6 – *Solution*

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If we let *x* represent the length of the tent pole, then the length of the rope, in terms of *x*, is $x\sqrt{2}$. It is also given as 10 feet. Therefore,

$$x\sqrt{2} = 10$$

$$x = \frac{10}{\sqrt{2}} = 5\sqrt{2}$$

The length of the tent pole is $5\sqrt{2}$ feet. Again, $5\sqrt{2}$ is the exact value of the length of the tent pole.

To find a decimal approximation, we replace $\sqrt{2}$ with 1.414 to obtain

$$5\sqrt{2} \approx 5(1.414) = 7.07 \text{ ft}$$