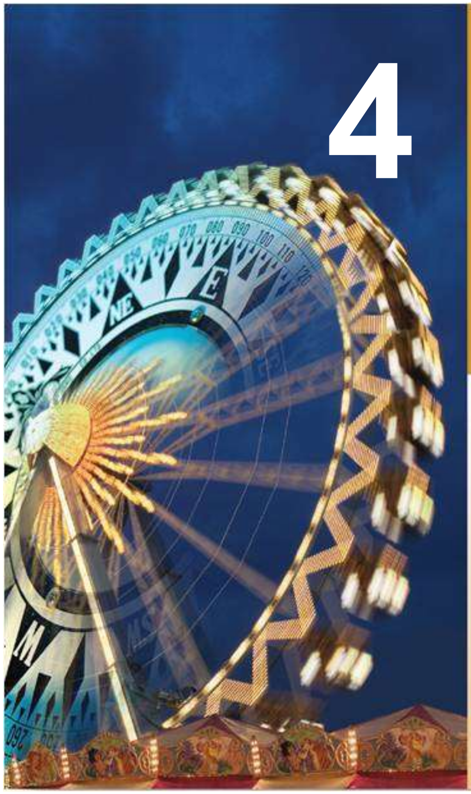

4 Graphing and Inverse Functions

SECTION 4.2

Amplitude, Reflection, and Period

Learning Objectives

1. Find the amplitude of a sine or cosine function.
2. Find the period of a sine or cosine function.
3. Graph a sine or cosine function having a different amplitude and period.
4. Solve a real-life problem involving a trigonometric function as a model.

Amplitude

Amplitude

First, we will consider the effect that multiplying a trigonometric function by a numerical factor has on the graph.

Example 1

Sketch the graph of $y = 2 \sin x$ for $0 \leq x \leq 2\pi$.

Solution:

The coefficient 2 on the right side of the equation will simply multiply each value of $\sin x$ by a factor of 2.

Therefore, the values of y in $y = 2 \sin x$ should all be twice the corresponding values of y in $y = \sin x$.

Example 1 – *Solution*

cont'd

Table 1 contains some values for $y = 2 \sin x$.

x	$y = 2 \sin x$	(x, y)
0	$y = 2 \sin 0 = 2(0) = 0$	$(0, 0)$
$\frac{\pi}{2}$	$y = 2 \sin \frac{\pi}{2} = 2(1) = 2$	$\left(\frac{\pi}{2}, 2\right)$
π	$y = 2 \sin \pi = 2(0) = 0$	$(\pi, 0)$
$\frac{3\pi}{2}$	$y = 2 \sin \frac{3\pi}{2} = 2(-1) = -2$	$\left(\frac{3\pi}{2}, -2\right)$
2π	$y = 2 \sin 2\pi = 2(0) = 0$	$(2\pi, 0)$

Table 1

Example 1 – *Solution*

cont'd

Figure 1 shows the graphs of $y = \sin x$ and $y = 2 \sin x$. (We are including the graph of $y = \sin x$ simply for reference and comparison. With both graphs to look at, it is easier to see what change is brought about by the coefficient 2.)

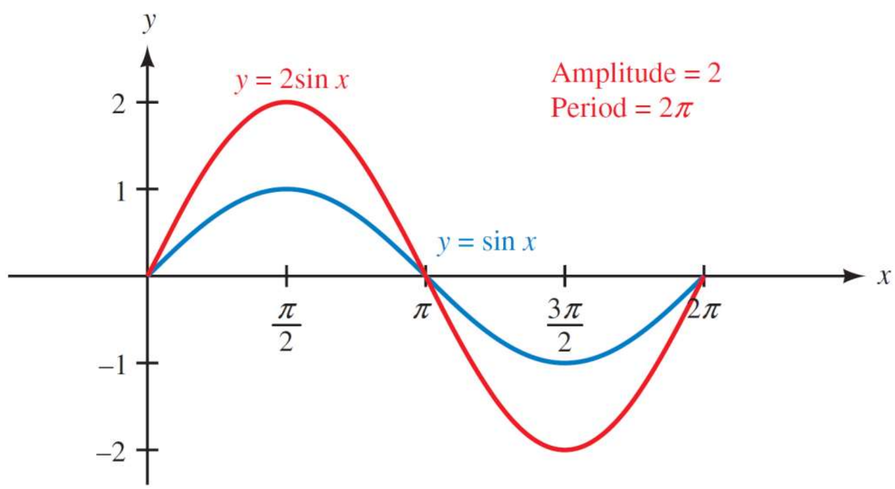

Figure 1

Example 1 – *Solution*

cont'd

The coefficient 2 in $y = 2 \sin x$ changes the amplitude from 1 to 2 but does not affect the period.

That is, we can think of the graph of $y = 2 \sin x$ as if it were the graph of $y = \sin x$ with the amplitude extended to 2 instead of 1.

Observe that the range has doubled from $[-1, 1]$ to $[-2, 2]$.

Amplitude

SUMMARY

Generalizing the results of these last two examples, we can say that if $A > 0$, then the graphs of $y = A\sin x$ and $y = A\cos x$ will have amplitude A and range $[-A, A]$.

Reflecting About the x-Axis

Reflecting About the x-Axis

In the previous example we only considered changes to the graph if the coefficient A was a positive number.

To see how a negative value of A affects the graph, we will consider the function $y = -2 \cos x$.

Example 3

Graph $y = -2 \cos x$, from $x = -2\pi$ to $x = 4\pi$.

Solution:

Each value of y on the graph of $y = -2 \cos x$ will be the opposite of the corresponding value of y on the graph of $y = 2 \cos x$.

The result is that the graph of $y = -2 \cos x$ is the reflection of the graph of $y = 2 \cos x$ about the x-axis.

Example 3 – *Solution*

cont'd

Figure 3 shows the extension of one complete cycle of $y = -2 \cos x$ to the interval $-2\pi \leq x \leq 4\pi$.

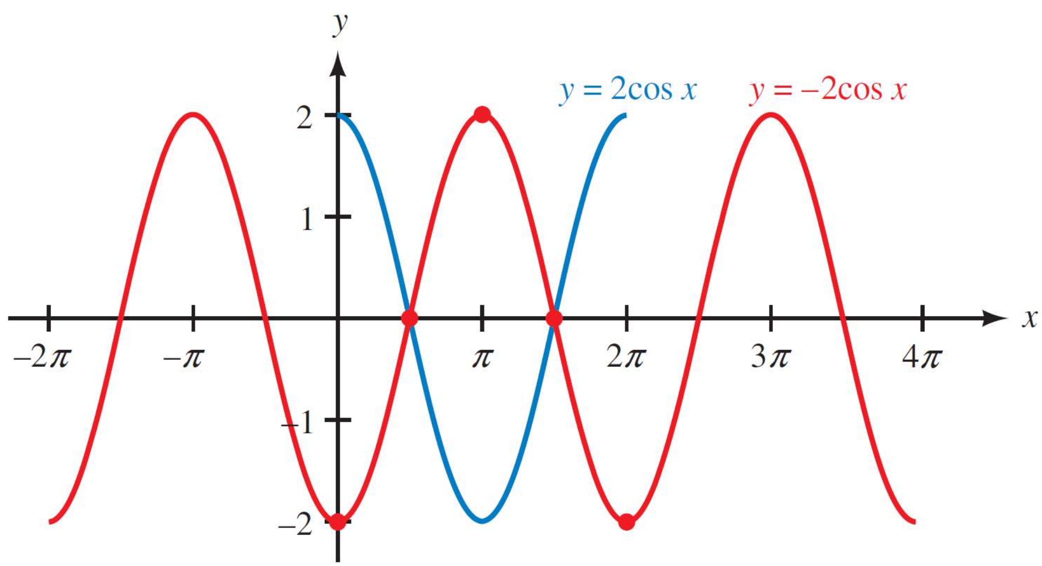

Figure 3

Reflecting About the *x*-Axis

SUMMARY

If $A < 0$, then the graphs of $y = A \sin x$ and $y = A \cos x$ will be sine and cosine graphs that have been reflected about the x-axis. The amplitude will be $|A|$.

Period

Period

Up to this point we have considered the effect that a coefficient, which multiplies the trigonometric function, has on the graph.

Now we will investigate what happens if we allow the input variable to have a coefficient.

Input is formally called the argument of the function.

Example 4

Graph $y = \sin 2x$ for $0 \le x \le 2\pi$.

Solution:

To see how the coefficient 2 in $y = \sin 2x$ affects the graph, we can make a table in which the values of x are multiples of $\pi/4$. (Multiples of $\pi/4$ are convenient because the coefficient 2 divides the 4 in $\pi/4$ exactly.)

Example 4 – *Solution*

cont'd

Table 3 shows the values of *x* and *y*, while Figure 4 contains the graphs of *y* = sin *x* and *y* = sin 2*x*.

x	$y = \sin 2x$	(x, y)
0	$y = \sin 2 \cdot 0 = \sin 0 = 0$	$(0, 0)$
$\frac{\pi}{4}$	$y = \sin 2 \cdot \frac{\pi}{4} = \sin \frac{\pi}{2} = 1$	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	$y = \sin 2 \cdot \frac{\pi}{2} = \sin \pi = 0$	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$y = \sin 2 \cdot \frac{3\pi}{4} = \sin \frac{3\pi}{2} = -1$	$\left(\frac{3\pi}{4}, -1\right)$
π	$y = \sin 2 \cdot \pi = \sin 2\pi = 0$	$(\pi, 0)$
$\frac{5\pi}{4}$	$y = \sin 2 \cdot \frac{5\pi}{4} = \sin \frac{5\pi}{2} = 1$	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	$y = \sin 2 \cdot \frac{3\pi}{2} = \sin 3\pi = 0$	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	$y = \sin 2 \cdot \frac{7\pi}{4} = \sin \frac{7\pi}{2} = -1$	$\left(\frac{7\pi}{4}, -1\right)$
2π	$y = \sin 2 \cdot 2\pi = \sin 4\pi = 0$	$(2\pi, 0)$

Table 3

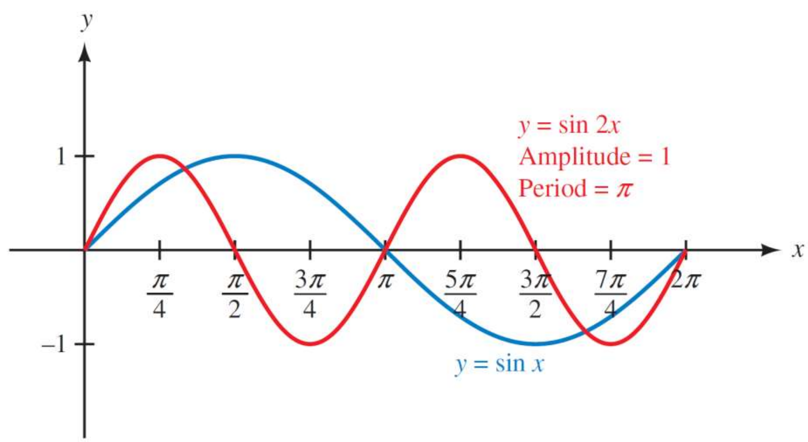

Figure 4

Example 4 – *Solution*

cont'd

The graph of $y = \sin 2x$ has a period of π. It goes through two complete cycles in 2π units on the x-axis. Notice that doubling the argument to the function has the reverse effect of halving the period.

This may be surprising at first, but we can see the reason for it by looking at a basic cycle.

We know that the sine function completes one cycle when the input value, or argument, varies between 0 and 2π.

Example 4 – *Solution*

cont'd

One cycle: $0 \le \text{argument} \le 2\pi$

$0 \le 2x \le 2\pi$ The argument is $2x$

$0 \le x \le \pi$ Divide by 2 to isolate x

Because of the factor of 2, the variable *x* only needs to reach π to complete one cycle, thus shortening the period.

Period

In general, for $y = \sin Bx$ or $y = \cos Bx$ to complete one cycle, the product Bx must vary from 0 to 2π.

Therefore

$$0 \le Bx \le 2\pi \qquad \text{if} \qquad 0 \le x \le \frac{2\pi}{B}$$

The period will be $2\pi/B$, and the graph will complete B cycles in 2π units.

Period

We summarize all of the information gathered from the previous example as follows.

AMPLITUDE AND PERIOD FOR SINE AND COSINE

If A is any real number and $B > 0$, then the graphs of $y = A \sin Bx$ and $y = A \cos Bx$ will have

$$\text{Amplitude} = |A| \qquad \text{and} \qquad \text{Period} = \frac{2\pi}{B}$$

In the situation where $B < 0$, we can use the properties of even and odd functions to rewrite the function so that B is positive.

Period

In the next example, we use this information about amplitude and period to graph one complete cycle of a sine and cosine curve and then extend these graphs to cover more than the one cycle.

We also take this opportunity to introduce a method of drawing the graph by constructing a “frame” for the basic cycle.

Example 7

Graph $y = 4\cos\left(-\frac{2}{3}x\right)$ for $-\frac{15\pi}{4} \leq x \leq \frac{15\pi}{4}$.

Solution:

Because cosine is an even function,

$$y = 4\cos\left(-\frac{2}{3}x\right) = 4\cos\left(\frac{2}{3}x\right)$$

Construct a Frame

The amplitude is 4, so $-4 \leq y \leq 4$.

Example 7 – *Solution*

We use the amplitude to determine the position of the upper and lower sides of a frame that will act as a boundary for a basic cycle.

Next we identify one complete cycle.

$$\text{One cycle:} \qquad 0 \le \frac{2}{3}x \le 2\pi$$

$$0 \le x \le 3\pi \qquad \text{Multiply by } \frac{3}{2} \text{ to isolate } x$$

The end points of the cycle give us the position of the left and right sides of the frame.

Example 7 – *Solution*

cont'd

Figure 7 shows how this is done.

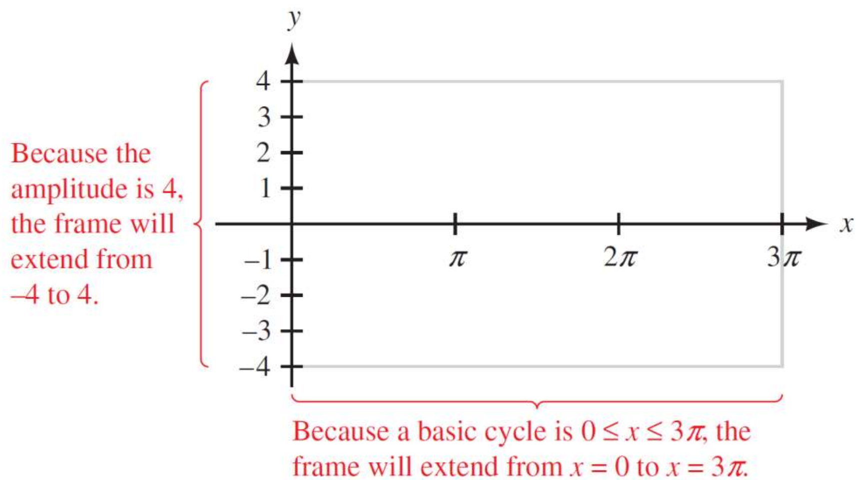

Figure 7

Subdivide the Frame

The period is 3π. Dividing by 4 gives us $3\pi/4$, so we will mark the *x*-axis in increments of $3\pi/4$.

Example 7 – *Solution*

cont'd

We already know where the cycle begins and ends, so we compute the three middle values:

$$1 \cdot \frac{3\pi}{4} = \frac{3\pi}{4}, \qquad 2 \cdot \frac{3\pi}{4} = \frac{3\pi}{2}, \qquad 3 \cdot \frac{3\pi}{4} = \frac{9\pi}{4}$$

We divide our frame in Figure 7 into four equal sections, marking the *x*-axis accordingly. Figure 8 shows the result.

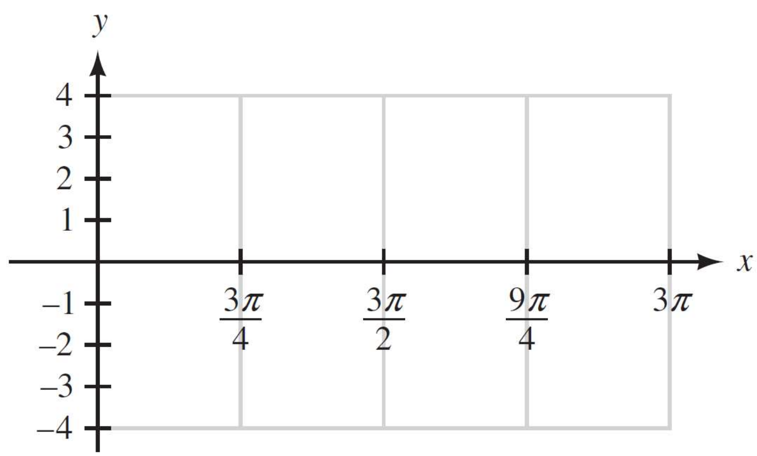

Figure 8

Example 7 – *Solution*

cont'd

Graph One Cycle

Now we use the frame to plot the key points that will define the shape of one complete cycle of the graph and then draw the graph itself (Figure 9).

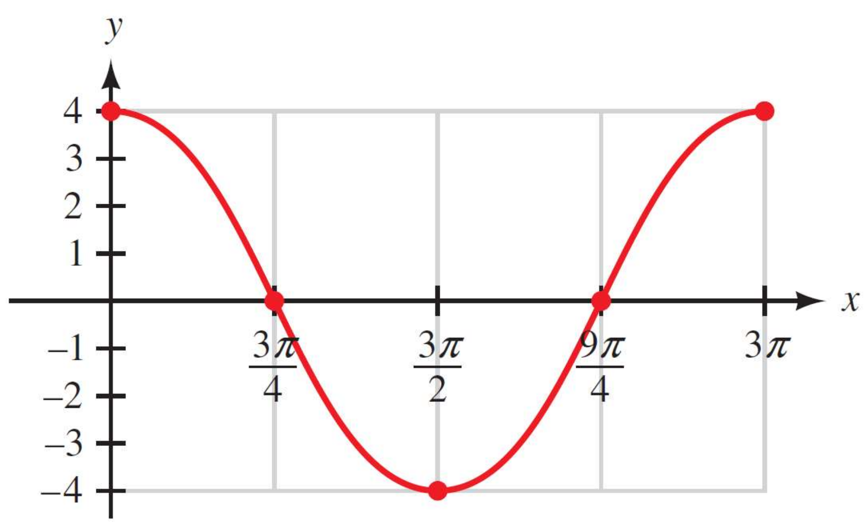

Figure 9

Example 7 – *Solution*

cont'd

Extend the Graph, if Necessary

The original problem asked for the graph on the interval

$$-\frac{15\pi}{4} \le x \le \frac{15\pi}{4}.$$

We extend the graph to the right by adding the first quarter of a second cycle.

On the left, we add another complete cycle (which takes the graph to -3π) and then add the last quarter of an additional cycle to reach $-15\pi/4$.

Example 7 – *Solution*

cont'd

The final graph is shown in Figure 10.

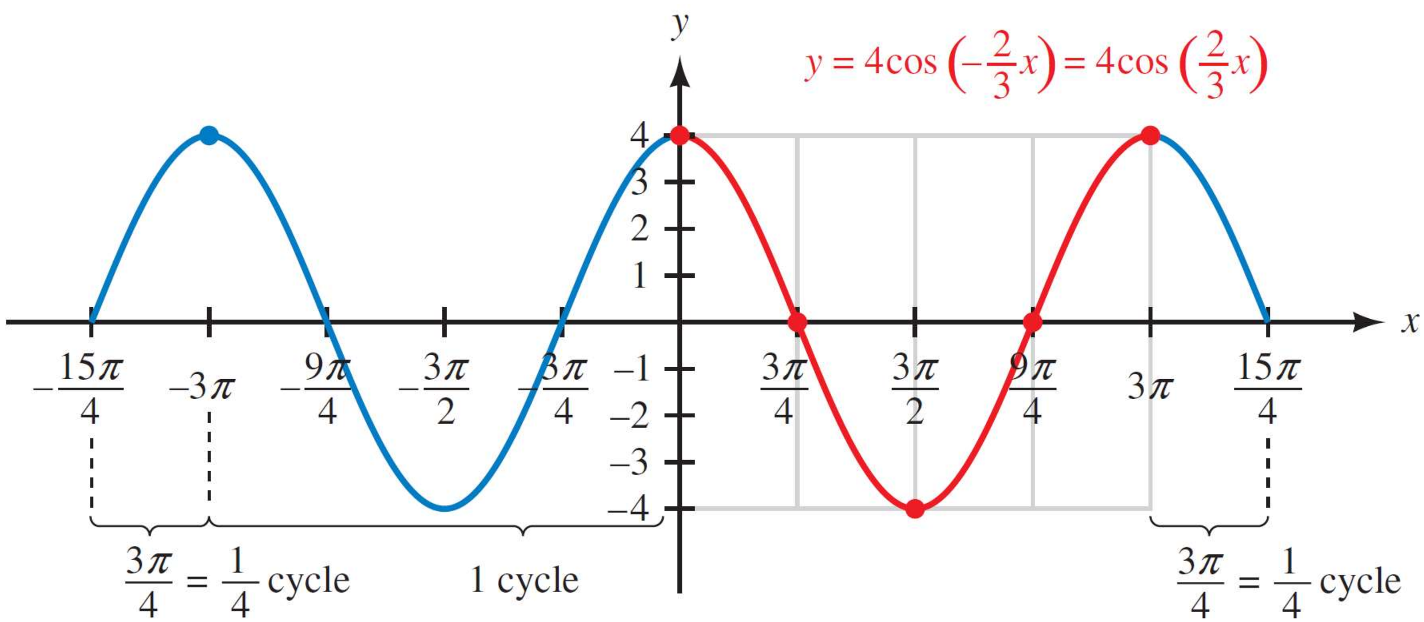

Figure 10